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Local null controllability of the N-dimensional Navier–Stokes system with nonlinear Navier-slip boundary conditions and $N-1$ scalar controls

Sergio Guerrero [a], Cristhian Montoya [b,*]

[a] *Sorbonne Université, UPMC Univ. Paris 6, UMR 7598 Laboratoire Jacques-Louis Lions, Paris, F-75005 France*
[b] *Departamento de Ingeniería Matemática, Universidad de Chile, Casilla 170/3 Correo 3, Santiago, Chile*

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A B S T R A C T

In this paper we deal with the local null controllability of the Navier–Stokes system with nonlinear Navier-slip boundary conditions and internal controls having one vanishing component.

R É S U M É

Dans cet article nous démonstrons la contrôlabilité exacte locale des équations de Navier–Stokes avec conditions aux limites non lin éaires de type Navier-slip et contrôles interne avec une composante nulle.

1. Introduction

Let Ω be a nonempty bounded connected open subset of $\mathbb{R}^N$ ($N = 2$ or $N = 3$) of class C^∞. Let $T > 0$ and let $\omega \subset \Omega$ be a (small) nonempty open subset which is the control domain. Here, we will use the notation $Q := \Omega \times (0, T)$, $\Sigma := \partial\Omega \times (0, T)$ and by $n(x)$ the outward unit normal vector to Ω at the point $x \in \partial\Omega$.

Let us consider the controlled Navier–Stokes system with nonlinear Navier slip boundary conditions. Given a nonlinear regular function $f : \mathbb{R}^N \to \mathbb{R}^N$ and an initial state y_0, we consider the following system:

$$\begin{cases} y_t - \nabla\cdot(Dy) + (y,\nabla)y + \nabla p = v\chi_\omega & \text{in} \quad Q, \\ \nabla\cdot y = 0 & \text{in} \quad Q, \\ y\cdot n = 0,\ (\sigma(y,p)\cdot n)_{tg} + f(y)_{tg} = 0 & \text{on} \quad \Sigma, \\ y(\cdot,0) = y_0(\cdot) & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail addresses: guerrero@ann.jussieu.fr (S. Guerrero), cmontoya@dim.uchile.cl (C. Montoya).

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where $v = v(x,t)$ stands for the control which acts in a arbitrary fixed domain $\omega \times (0,T)$ and $supp\, v \subset \omega \times (0,T)$, χ_ω is a smooth positive function such that $\chi_\omega = 1$ in ω', where $\omega' \Subset \omega$. Respect to the boundary conditions, we mention that in 1823, C.L. Navier (see [21]) established a slip-with-friction boundary condition and claimed that the component of the fluid velocity tangential to the surface should be proportional to the rate of strain at the surface. The velocity's component normal to the surface is naturally zero as mass is not able to penetrate an impermeable solid surface [21]. This can be expressed by

$$y \cdot n = 0 \quad \text{and} \quad (\sigma(y,p) \cdot n)_{tg} + k y_{tg} = 0 \quad \text{on } \Sigma,$$

where $\sigma(y,p) := -pId + Dy$ is the stress tensor, D is the symmetrized gradient of y, p is the pressure, Id is the identity matrix, tg stands for the tangential component of the corresponding vector field, i.e., (see [21]):

$$y_{tg} = y - (y \cdot n)n$$

and k is a scalar friction function that measures the local viscous coupling between fluid and solid.

Physically a nonzero slip length arises from the unequal wall and fluid densities, the weak wall–fluid interaction and the high temperature. Although in most of the situations, the Navier-slip boundary condition can be reduced to the no-slip boundary conditions due to extremely small slip length. However, in some cases as in the driven cavity flow problem, aerodynamics processes, weather forecast, turbulence problems, among others, it has been shown that the Navier-slip boundary condition is valid and removes un-physical singularities (see [5] and references therein). Then, the theoretical analysis is complicated as well as numerical solutions of the model and an alternative is then to reduce the no-slip condition on rough boundaries to *ad hoc* boundary conditions, the so-called *wall laws*, on a smooth domain.

Let us point out that our boundary conditions corresponds to a law of the wall that appear in turbulent flows, specifically when k may not depend on $|y|$ linearly. We invite to the interested reader to see [5], [19] for a complete discussion on this subject.

In the context of controllability, the papers by Coron [7] and Imanuvilov [17] show results of the approximate controllability and local exact controllability for the Navier–Stokes system with Navier-slip boundary conditions in two dimensions, with some restrictions in each case. The system (1.1) has been studied by Guerrero [16], in this paper the author proved the local null controllability to the trajectories of (1.1) in dimension N using Carleman estimates for the associated linear system and fixed point arguments. On the other hand, recent papers by Coron and Guerrero [6], Carreño and Guerrero [4] are evidence of the null controllability and local null controllability of the Navier–Stokes system with $N-1$ scalar controls, even thought they use homogeneous Dirichlet boundary conditions. We also highlight the work by Coron and Lissy [8], whose authors have proved the local null controllability of the 3D Navier–Stokes system with one scalar control. Finally, we refer to the more recent work on global exact controllability of the Navier–Stokes equation with Navier slip-with-friction boundary conditions [9], where the authors have used the return method, asymptotic convergence and dissipation estimates for the boundary layer in order to prove the main result.

The main objective of this paper is to obtain the local null controllability of system (1.1) by means of $N-1$ scalar controls, see Theorem 1.1.

Let us now introduce several spaces which are usual in the context of problems modeling incompressible fluids:

$$V := \{u \in H_0^1(\Omega)^N : \nabla \cdot u = 0 \text{ in } \Omega\},$$

$$H := \{u \in L^2(\Omega)^N : \nabla \cdot u = 0, \text{ in } \Omega \;\; u \cdot n = 0 \text{ on } \partial\Omega\}$$

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and

$$W = \{u \in H^1(\Omega)^N : \nabla \cdot u = 0 \text{ in } \Omega, \ \ u \cdot n = 0 \text{ on } \partial\Omega\}.$$

Our main result is given in the following theorem.

Theorem 1.1. *Let us assume that $i \in \{1, \dots, N\}$ and $f \in C^4(\mathbb{R}^N; \mathbb{R}^N)$ with $f(0) = 0$. Then, for every $T > 0$ and $\omega \subset \Omega$, there exists $\delta > 0$ such that, for every $y_0 \in H^3(\Omega)^N \cap W$ satisfying $\|y_0\|_{H^3(\Omega)^N \cap W} \leq \delta$ and the compatibility condition*

$$(Dy_0 \cdot n)_{tg} + (f(y_0))_{tg} = 0 \ \text{on} \ \partial\Omega, \tag{1.2}$$

we can find a control

$$v \in L^2(0, T; H^2(\omega)^N) \cap H^1(0, T; L^2(\omega)^N),$$

with $v_i \equiv 0$ and an associated solution (y, p) to (1.1) verifying $y(\cdot, T) = 0$ in Ω.

To prove Theorem 1.1, we first deduce a null controllability result for a linearized system around zero associated to (1.1):

$$\begin{cases} y_t - \nabla \cdot (Dy) + \nabla p = h + v\chi_\omega & \text{in} \quad Q, \\ \nabla \cdot y = 0 & \text{in} \quad Q, \\ y \cdot n = 0, \ (\sigma(y, p) \cdot n)_{tg} + (A(x, t)y)_{tg} = 0 & \text{on} \quad \Sigma, \\ y(\cdot, 0) = y_0(\cdot) & \text{in} \quad \Omega, \end{cases} \tag{1.3}$$

where A is a $N \times N$ matrix-valued function in a suitable space and h decreases exponentially to zero in T. Finally, we apply Kakutani's fixed point theorem and an inverse mapping theorem to conclude the local null controllability for the nonlinear system (1.1).

The paper is organized as follows. In Section 2, we present a previous regularity result proved in [16] and other that we prove here for systems as (1.3). In section 3, we establish a Carleman inequality needed to deal with the controllability problems. In section 4, we prove the null controllability of the linear system (1.3). Finally, in Section 5 we give the proof of Theorem 1.1 using fixed point arguments.

Before starting with Section 2, we consider several Hilbert spaces for $\varepsilon > 0$ small enough:

$$\begin{gathered} P^0_\varepsilon := H^{1/2+\varepsilon}(0, T; H^{1+\varepsilon}(\partial\Omega)^{N\times N}), \quad P^1_\varepsilon := H^{5/4+\varepsilon}(0, T; L^2(\partial\Omega)^{N\times N}), \\ P^2 := L^2(0, T; H^{5/2}(\partial\Omega)^{N\times N}), \\ Z_\varepsilon := H^{5/4+\varepsilon}(0, T; H^1(\Omega)^N \cap W) \cap L^2(0, T; H^3(\Omega)^N \cap W) \end{gathered} \tag{1.4}$$

and

$$Y_1 := L^2(0, T; H^2(\Omega)^N) \cap H^1(0, T; L^2(\Omega)^N), \quad Y_2 := L^2(0, T; H^4(\Omega)^N) \cap H^2(0, T; L^2(\Omega)^N).$$

2. Preliminary results

In order to prove the main theorem of this paper, we introduce some preliminary results which will be used later on. More precisely, we present regularity results concerning the Stokes system with linear Navier-slip boundary conditions.

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The proof of the following result can be found in [16].

Lemma 2.1. *Let* $A \in P^0_\varepsilon$, $u_0 \in H$, $f_0 \in L^2(0,T;W')$, $f_2 \in L^2(0,T;H^{-1/2}(\partial\Omega)^N)$ *and let* u *be the weak solution of the system*

$$\begin{cases} u_t - \nabla\cdot(Du) + \nabla\theta = f_0 & in \quad Q, \\ \nabla\cdot u = 0 & in \quad Q, \\ u\cdot n = 0,\ (\sigma(u,\theta)\cdot n)_{tg} + (A(x,t)u)_{tg} = f_2 & on \quad \Sigma, \\ u(\cdot,0) = u_0(\cdot) & in \quad \Omega, \end{cases} \tag{2.1}$$

namely, the function u *satisfying*

$$\begin{cases} \int\limits_\Omega u_t(t)\cdot v dx + \frac{1}{2}\int\limits_\Omega Du(t) : Dv dx + \int\limits_{\partial\Omega} Au(t)\cdot v d\sigma \\ = \int\limits_\Omega f_0(t)\cdot v dx + \int\limits_{\partial\Omega} f_2(t)\cdot v d\sigma \quad a.e.\ t\in(0,T), \quad \forall v\in W, \\ u(\cdot,0) = u_0(\cdot) \quad in\ \ \Omega. \end{cases}$$

Then, if we further assume $u_0 \in W$ *and*

$$f_0 \in L^2(Q)^N,\ f_2 \in L^2(0,T;H^{1/2}(\partial\Omega)^N),\ f_2 \in H^{1/4+\varepsilon}(0,T;H^{-\varepsilon}(\partial\Omega)^N),$$

u *is actually, together with a pressure* θ, *the strong solution of* (2.1), *i.e.,* $(u,\theta) \in Y_1 \times L^2(0,T;H^1(\Omega))$. *Furthermore, there exists a positive constant* C *such that*

$$\begin{aligned} \|u\|_{Y_1} + \|\theta\|_{L^2(0,T;H^1(\Omega))} \le Ce^{CT\|A\|^2_{P^0_\varepsilon}}(1+\|A\|^2_{P^0_\varepsilon})\big(\|f_0\|_{L^2(Q)^N} \\ + \|f_2\|_{L^2(0,T;H^{1/2}(\partial\Omega)^N)} + \|f_2\|_{H^{1/4+\varepsilon}(0,T;H^{-\varepsilon}(\partial\Omega)^N)} + \|u_0\|_{H^1(\Omega)^N}\big). \end{aligned} \tag{2.2}$$

Remark 2.1. The author in [16] proved Lemma 2.1 whenever

$$A \in H^{1-\ell}(0,T;W^{\nu_1,\nu_1+1}(\partial\Omega)^{N\times N}),$$

where $0<\ell<1/2$ is arbitrarily close to $1/2$ and $\nu_1>1$ is arbitrarily close to 1. Observe that this hypothesis is satisfied if $A\in P^0_\varepsilon$.

Using the above Lemma, we prove now a regularity result for the solution of (2.1). To this end, we will impose the following compatibility condition:

$$(Du_0\cdot n)_{tg} + (A(\cdot,0)u_0)_{tg} = f_2(\cdot,0) \quad \text{on } \partial\Omega. \tag{2.3}$$

Theorem 2.1. *Let* $A \in P^1_\varepsilon \cap P^2$, $u_0 \in H^3(\Omega)^N \cap W$ *satisfying* (2.3), $f_0 \in Y_1$, $f_2 \in L^2(0,T;H^{5/2}(\partial\Omega)^N) \cap H^1(0,T;H^{1/2}(\partial\Omega)^N)$, *and let* u *be the strong solution of system*

$$\begin{cases} u_t - \nabla\cdot(Du) + \nabla\theta = f_0 & in \quad Q, \\ \nabla\cdot u = 0 & in \quad Q, \\ u\cdot n = 0,\ (\sigma(u,\theta)\cdot n)_{tg} + (A(x,t)u)_{tg} = f_2 & on \quad \Sigma, \\ u(\cdot,0) = u_0(\cdot) & in \quad \Omega. \end{cases} \tag{2.4}$$

Then, $(u,\theta) \in Y_2 \times L^2(0,T;H^3(\Omega))$ *and there exists a positive constant* C *such that*

$$\begin{aligned} &\|u\|_{Y_2} + \|\theta\|_{L^2(0,T;H^3(\Omega))} \\ &\leq C(A)\Big(\|f_0\|_{Y_1} + \|f_2\|_{L^2(0,T;H^{5/2}(\partial\Omega)^N)} + \|f_2\|_{H^1(0,T;H^{1/2}(\partial\Omega)^N)} + \|u_0\|_{H^3(\Omega)^N}\Big), \end{aligned} \tag{2.5}$$

where

$$C(A) = Ce^{CT\|A\|^2_{P^0_\varepsilon}}\Big(1 + \|A\|^2_{P^0_\varepsilon}\Big)\Big[1 + \|A\|^3_{P^1_\varepsilon} + \|A\|^3_{P^2}\Big]. \tag{2.6}$$

Proof of Theorem 2.1. We consider (2.4) like a stationary system, that is to say:

$$\begin{cases} -\nabla\cdot(Du) + \nabla\theta = f_0 - u_t & \text{in} \quad \Omega, \\ \nabla\cdot u = 0 & \text{in} \quad \Omega, \\ u\cdot n = 0,\ (\sigma(u,\theta)\cdot n)_{tg} + (A(x,t))_{tg} = f_2 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.7}$$

for almost every $t \in (0,T)$.

The rest of the proof is divided in two steps.

Step 1. The goal will be to prove that the weak solution (u,θ) of the stationary system

$$\begin{cases} -\nabla\cdot(Du) + \nabla\theta = g_0 & \text{in} \quad \Omega, \\ \nabla\cdot u = g_1 & \text{in} \quad \Omega, \\ u\cdot n = 0,\ (\sigma(u,\theta)\cdot n)_{tg} = g_2 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.8}$$

actually belongs to $H^3(\Omega)^N \times H^2(\Omega)$, whenever $g_0 \in H^1(\Omega)^N, g_1 \in H^2(\Omega)$ and $g_2 \in H^{3/2}(\partial\Omega)^N$.

In accordance with estimate (2.2) for the stationary case and for $A = 0$, we obtain that the weak solution of (2.8) satisfies

$$\|u\|_{H^2(\Omega)^N} + \|\theta\|_{H^1(\Omega)} \leq C\Big(\|g_0\|_{L^2(\Omega)^N} + \|g_1\|_{H^1(\Omega)} + \|g_2\|_{H^{1/2}(\partial\Omega)^N}\Big), \tag{2.9}$$

for a positive constant C.

The interior regularity readily follows from the corresponding result with homogeneous Dirichlet boundary conditions, which can be found in [22], for instance. Then, for every $\Omega' \subset\subset \Omega$, we have $u \in H^3(\Omega')^N$, $\theta \in H^2(\Omega')$ and

$$\|u\|_{H^3(\Omega')^N} + \|\theta\|_{H^2(\Omega')} \leq C\Big(\|g_0\|_{H^1(\Omega)^N} + \|g_1\|_{H^2(\Omega)} + \|g_2\|_{H^{1/2}(\partial\Omega)^N}\Big), \tag{2.10}$$

for some positive constant $C(\Omega',\Omega)$.

In order to obtain this close to the boundary, we consider $x_0 \in \partial\Omega$ and U_0 a neighborhood of x_0. Then, it suffices to prove that $u \in H^3(\Omega\cap\tilde U)^N$ and $\theta \in H^2(\Omega\cap\tilde U)$, for every $\tilde U \subset\subset U_0$.

To this end, let ψ be a $W^{3,\infty}$ diffeomorphism which sends the set

$$C_0 := \{(\xi',\xi_N) \in \mathbb{R}^N : |\xi_i| < \alpha_0 \quad i = 1,\cdots,N-1, |\xi_N| < \beta_0\}$$

onto U_0 and which verifies

$$\psi(C_0^+) = \Omega\cap U_0, \quad \psi(\Delta_{\alpha_0}) = \partial\Omega\cap U_0,$$

where we have denoted $C_0^+ = C_0 \cap \mathbb{R}^N_+$ and $\Delta_{\alpha_0} = \partial\mathbb{R}^N_+ \cap C_0$. Let us now introduce a cut-off function $\zeta \in C^2(U_0)$ such that

$$\zeta \equiv 1 \text{ in } \tilde{U} \quad \text{and} \quad \operatorname{supp} \zeta \subset U_1 \subset\subset U_0, \tag{2.11}$$

where U_1 is a regular open set. Then, let us set $z = \zeta u$, $h = \zeta\theta$. They verify:

$$\begin{cases} -\nabla\cdot(Dz) + \nabla h = g_0^* & \text{in} \quad \Omega\cap U_0, \\ \nabla\cdot z = g_1^* & \text{in} \quad \Omega\cap U_0, \\ z\cdot n = 0,\ (\sigma(z,h)\cdot n)_{tg} = g_2^* & \text{on} \quad \partial\Omega\cap U_0, \\ z = 0 & \text{on} \quad \Omega\cap\partial U_0, \end{cases} \tag{2.12}$$

with

$$\begin{aligned} &g_0^* = \zeta g_0 - 2\nabla\zeta\cdot\nabla u - \nabla\zeta\cdot\nabla^t u - \Delta\zeta u - \nabla\nabla\zeta\cdot u + \theta\nabla\zeta - g_1\nabla\zeta \in H^1(\Omega\cap U_0)^N, \\ &g_1^* = \zeta g_1 + \nabla\zeta\cdot u \in H^2(\Omega\cap U_0) \quad \text{and} \quad g_2^* = \zeta g_2 + \frac{\partial\zeta}{\partial n} u \in H^{3/2}(\partial\Omega\cap U_0)^N. \end{aligned} \tag{2.13}$$

Let us now perform the change of variable $x = \psi(\xi)$. If we define $\tilde{z} = z\circ\psi$, $\tilde{h} = h\circ\psi$ and $\tilde{n} = n\circ\psi$, then

$$\frac{\partial}{\partial x_i} z_s = \sum_{k=1}^N \frac{\partial \tilde{z}_s}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_i} = \nabla\tilde{z}_s\cdot\nabla_i\psi^{-1}, \quad \forall s = 1,\dots,N,$$

where we have denoted $\nabla_i\psi^{-1}$ the ith-column of $\nabla\psi^{-1}$. Observe that

$$\frac{\partial}{\partial x_l}\Big(\frac{\partial}{\partial x_i} z_s\Big) = \sum_{j,k=1}^N \Big(\frac{\partial^2\tilde{z}_s}{\partial\xi_k\partial\xi_j}\frac{\partial\xi_j}{\partial x_l}\frac{\partial\xi_k}{\partial x_i}\Big) + \sum_{k=1}^N \frac{\partial\tilde{z}_s}{\partial\xi_k}\frac{\partial^2\xi_k}{\partial x_l\partial x_i}.$$

Therefore

$$\begin{aligned} \Delta z_s = &\sum_{i,j,k=1}^N \Big(\frac{\partial^2\tilde{z}_s}{\partial\xi_k\partial\xi_j}\frac{\partial\xi_j}{\partial x_i}\frac{\partial\xi_k}{\partial x_i}\Big) + \sum_{k,i=1}^N \frac{\partial\tilde{z}_s}{\partial\xi_k}\frac{\partial^2\xi_k}{\partial x_i^2} = Hess(\tilde{z}_s) : \Big(\sum_{i=1}^N \frac{\partial\xi_j}{\partial x_i}\frac{\partial\xi_k}{\partial x_i}\Big)_{j,k} + \sum_{k=1}^N \frac{\partial\tilde{z}_s}{\partial\xi_k}\Delta\xi_k \\ &= Hess(\tilde{z}_s) : \nabla\psi^{-1}\nabla^t\psi^{-1} + \nabla\tilde{z}_s\cdot\Delta\psi^{-1}, \end{aligned}$$

where $Hess(\tilde{z}_s)$ represents the Hessian matrix on $\tilde{z}_s$ and $\Delta\psi^{-1} := \Delta\xi := (\Delta\xi_1,\dots,\Delta\xi_N)$. Moreover,

$$div\, z = \sum_{s,j=1}^N \frac{\partial\tilde{z}_s}{\partial\xi_j}\frac{\partial\xi_j}{\partial x_s} = \nabla\tilde{z} : \nabla^t\psi^{-1} \quad \text{and} \quad \frac{\partial}{\partial x_s} h = \sum_{j=1}^N \frac{\partial\tilde{h}}{\partial\xi_j}\frac{\partial\xi_j}{\partial x_s} = \nabla\tilde{h}\cdot\nabla_s\psi^{-1}.$$

Then, taking into account that for every $i = 1,\dots,N$ we have

$$(\nabla\cdot Dz)_i = \Delta z_i + \partial_i\, div\, z,$$

we find from (2.12) that $\tilde{z}_i$ satisfies the following system for $i = 1,\dots,N$:

$$\begin{cases} -Hess(\tilde{z}_i) : \nabla\psi^{-1}\nabla^t\psi^{-1} - \nabla\tilde{z}_i\cdot\Delta\psi^{-1} + \nabla\tilde{h}\cdot\nabla_i\psi^{-1} = (\tilde{g_0}^*)_i + \partial_i\tilde{g_1}^* & \text{in} \quad C_0^+, \\ \nabla\tilde{z} : \nabla^t\psi^{-1} = \tilde{g_1}^* & \text{in} \quad C_0^+, \\ \tilde{z}\cdot\tilde{n} = 0, \quad (\tilde{\sigma}(\tilde{z})\cdot\tilde{n})_{tg} = \tilde{g_2}^* & \text{on} \quad \partial\mathbb{R}^N_+\cap C_0, \\ \tilde{z} = 0 & \text{on} \quad \partial C_0^+\cap\mathbb{R}^N_+, \end{cases} \tag{2.14}$$

where we have denoted

$$\tilde{g_0}^* = g_0^* \circ \psi, \quad \tilde{g_1}^* = g_1^* \circ \psi, \quad \tilde{g_2}^* = g_2^* \circ \psi$$

and

$$(\tilde{\sigma}(\tilde{z}))_{is} := \nabla \tilde{z}_s \cdot \nabla_i \psi^{-1} + \nabla \tilde{z}_i \cdot \nabla_s \psi^{-1}, \ \forall 1 \leq i, s \leq N.$$

On the other hand, note that for every function F in $H^\ell(\Omega)$ ($\ell \in \mathbb{N}, \ell \leq 3$), $\tilde{F} = F \circ \psi$ belongs to $H^\ell(C_0^+)$ and there exists a positive constant $C = C(\Omega)$ such that

$$\|\tilde{F}\|_{H^\ell(C_0^+)} \leq C \|F\|_{H^\ell(\Omega)}.$$

Now, observe that $\tilde{z} \in \tilde{X}_{0,2}$, with

$$\tilde{X}_{0,2} := \{\tilde{z} \in H^2(C_0^+)^N : \tilde{z} = 0 \text{ on } \partial C_0^+ \cap \mathbb{R}^N_+, \ \tilde{z} \cdot \tilde{n} = 0 \text{ on } \partial \mathbb{R}^N_+ \cap C_0\}.$$

Let us introduce $C_1 = \psi(U_1)$ (recall that $U_1 \subset\subset U_0$) and $d = dist(\partial C_0^+, \partial C_1^+)$. Then, we have $\delta^k_m \tilde{z} \in \tilde{X}_{0,2}$ for any $1 \leq k \leq N-1$ and any $|m| < d/2$, where we have denoted

$$\tilde{X}_{1,2} := \{\tilde{z} \in H^2(C_1^+)^N : \tilde{z} = 0 \text{ on } \partial C_1^+ \cap \mathbb{R}^N_+, \ \tilde{z} \cdot \tilde{n} = 0 \text{ on } \partial \mathbb{R}^N_+ \cap C_1\},$$

and (see [2])

$$\delta^k_m(f) := \tau^k_m(f) - f, \quad \tau^k_m(f) = (\xi \to f(\xi + m e_k)) \tag{2.15}$$

(see (2.9) and (2.11)). We denote now $\tilde{w} = \delta^k_m \tilde{z}$, $\tilde{\pi} = \delta^k_m \tilde{h}$. We have:

$$\begin{gathered}
\delta^k_m(Hess(\tilde{z}_i) : \nabla\psi^{-1}\nabla^t\psi^{-1}) = Hess(\tilde{w}_i) : \nabla\psi^{-1}\nabla^t\psi^{-1} + Hess(\tilde{z}_i(\xi + m e_k)) : \delta^k_m(\nabla\psi^{-1}\nabla^t\psi^{-1}). \\
\delta^k_m(\nabla \tilde{z}_i \cdot \Delta\psi^{-1}) = \nabla \tilde{w}_i \cdot \Delta\psi^{-1} + \nabla \tilde{z}_i(\xi + m e_k) \cdot \delta^k_m(\Delta\psi^{-1}). \\
\delta^k_m(\nabla \tilde{z} : \nabla^t\psi^{-1}) = \nabla \tilde{w} : \nabla^t\psi^{-1} + \nabla \tilde{z}(\xi + m e_k) : \delta^k_m \nabla^t\psi^{-1}
\end{gathered}$$

and

$$\delta^k_m(\nabla \tilde{h} \cdot \nabla_i\psi^{-1}) = \nabla \tilde{\pi} \cdot \nabla_i\psi^{-1} + \nabla \tilde{h}(\xi + m e_k) \cdot \delta^k_m \nabla_i\psi^{-1}.$$

Additionally,

$$\delta^k_m(\tilde{z} \cdot \tilde{n}) = \tilde{w} \cdot \tilde{n}$$

and

$$\delta^k_m((\sigma(\tilde{z}, \tilde{h}) \cdot \tilde{n})_{tg}) = (\tilde{\sigma}(\tilde{w}) \cdot \tilde{n})_{tg} + \left[\sum_{s=1}^N (\nabla \tilde{z}_s(\xi + m e_k) \cdot \delta^k_m \nabla_i\psi^{-1} + \nabla \tilde{z}_i(\xi + m e_k) \cdot \delta^k_m \nabla_s\psi^{-1})\tilde{n}_s\right]_{tg}$$

on $\partial \mathbb{R}^N_+ \cap C_1$. The last two identities readily follow from (2.15) and the fact that $\tilde{n}_j(\xi + m e_k) = \tilde{n}_j(\xi)$ on $C_1 \cap \partial \mathbb{R}^N_+$, for every $k = 1, \ldots, N-1$ and for every $j = 1, \ldots, N$. Taking into account the above identities and (2.14), the pair $(\tilde{w}, \tilde{\pi})$ satisfies:

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$$\begin{cases} -Hess(\tilde{w}_i) : \nabla\psi^{-1}\nabla^t\psi^{-1} - \nabla\tilde{w}_i \cdot \Delta\psi^{-1} + \nabla\tilde{\pi} \cdot \nabla_i\psi^{-1} = G_{0,i} + \partial_i G_1 & \text{in} \quad C_1^+, \\ \nabla\tilde{w} : \nabla^t\psi^{-1} = G_1 & \text{in} \quad C_1^+, \\ \tilde{w}\cdot\tilde{n} = 0, \quad (\tilde{\sigma}(\tilde{w})\cdot\tilde{n})_{tg} = G_2 & \text{on} \quad \partial\mathbb{R}^N_+ \cap C_1, \end{cases} \tag{2.16}$$

where

$$\begin{aligned} G_{0,i} &= \delta^k_m(\tilde{g_0}^*)_i + Hess(\tilde{z}_i(\xi+me_k)) : \delta^k_m(\nabla\psi^{-1}\nabla^t\psi^{-1}) + \nabla\tilde{z}_i(\xi+me_k)\cdot\delta^k_m\Delta\psi^{-1} \\ &\quad + \partial_i(\nabla\tilde{z}(\xi+me_k) : \delta^k_m\nabla^t\psi^{-1}) - \nabla\tilde{h}(\xi+me_k)\cdot\delta^k_m\nabla_i\psi^{-1}, \\ G_1 &= \delta^k_m(\tilde{g_1}^*) - \nabla\tilde{z}(\xi+me_k) : \delta^k_m\nabla^t\psi^{-1}, \\ G_2 &= \delta^k_m(\tilde{g_2}^*) - \left[\sum_{s=1}^N(\nabla\tilde{z}_s(\xi+me_k)\cdot\delta^k_m\nabla_i\psi^{-1} + \nabla\tilde{z}_i(\xi+me_k)\cdot\delta^k_m\nabla_s\psi^{-1})\tilde{n}_s\right]_{tg}. \end{aligned}$$

Let us now estimate $G_{0,i}$ in the $L^2(C_1^+)$ norm. We have

$$\begin{aligned} \|\delta^k_m(\tilde{g_0}^*)_i\|_{L^2(C_1^+)} &\leq C|m|\|\nabla(\tilde{g_0}^*)_i\|_{L^2(C_1^+)} \leq C|m|\|(\tilde{g_0}^*)_i\|_{H^1(C_1^+)}, \\ \|Hess(\tilde{z}_i(\xi+me_k)) : \delta^k_m(\nabla\psi^{-1}\nabla^t\psi^{-1})\|_{L^2(C_1^+)} &\leq C|m|\|\tilde{z}\|_{H^2(C_1^+)^N}, \\ \|\nabla\tilde{z}_i(\xi+me_k)\cdot\delta^k_m\Delta\psi^{-1}\|_{L^2(C_1^+)} &\leq C(k,\Omega)|m|\|\nabla\tilde{z}_i\|_{L^2(C_1^+)^N}, \\ \|\partial_i(\nabla\tilde{z}(\xi+me_k) : \delta^k_m\nabla^t\psi^{-1})\|_{L^2(C_1^+)} &\leq C|m|\|\tilde{z}\|_{H^2(C_1^+)^N} \end{aligned}$$

and

$$\|\nabla\tilde{h}(\xi+me_k)\cdot\delta^k_m\nabla_i\psi^{-1}\|_{L^2(C_1^+)} \leq C|m|\|\nabla\tilde{h}\|_{L^2(C_1^+)^N}.$$

Therefore

$$\|G_0\|_{L^2(C_1^+)^N} \leq C|m|\Big(\|g_0^*\|_{H^1(\Omega)^N} + \|z\|_{H^2(\Omega)^N} + \|\nabla h\|_{L^2(\Omega)}\Big).$$

In the same way we can estimate G_1 in $H^1(C_1^+)$ from

$$\|\delta^k_m(\tilde{g_1}^*)\|_{H^1(C_1^+)} \leq |m|\|\tilde{g_1}^*\|_{H^2(C_1^+)}$$

and we obtain

$$\|G_1\|_{H^1(C_1^+)} \leq C|m|\Big(\|g_1^*\|_{H^2(\Omega)} + \|z\|_{H^2(\Omega)^N}\Big).$$

Finally, for G_2 we get

$$\|G_2\|_{H^{1/2}(\partial\mathbb{R}^N_+\cap C_1)^N} \leq C|m|(\|\tilde{g}_2^*\|_{H^{3/2}(\partial\mathbb{R}^N_+\cap C_1)^N} + \|\tilde{z}\|_{H^{3/2}(\partial\mathbb{R}^N_+\cap C_1)^N}).$$

Then, using the definition of g_i^* $(i = 0, 1, 2)$ given in (2.13) and the estimate (2.9) for the solutions of the stationary problems (2.8) and (2.12), we obtain

$$\|G_0\|_{L^2(C_1^+)^N} \leq C|m|\Big(\|g_0\|_{H^1(\Omega)^N} + \|g_1\|_{H^1(\Omega)} + \|g_2\|_{H^{1/2}(\partial\Omega)^N}\Big),$$

$$\|G_1\|_{H^1(C_1^+)} \leq C|m|\Big(\|g_0\|_{L^2(\Omega)^N} + \|g_1\|_{H^2(\Omega)} + \|g_2\|_{H^{1/2}(\partial\Omega)^N}\Big)$$

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and

$$\|G_2\|_{H^{1/2}(\partial\mathbb{R}^N_+\cap C_1)^N} \leq C|m|\Big(\|g_2\|_{H^{3/2}(\partial\Omega)^N} + \|g_0\|_{L^2(\Omega)^N} + \|g_1\|_{H^1(\Omega)}\Big).$$

In consequence, the solution of (2.16) belongs to $\tilde{X}_{1,2} \times H^1(C_1^+)$ and satisfies

$$\|\delta^k_m \tilde{z}\|_{H^2(C_1^+)^N} + \|\delta^k_m \tilde{h}\|_{H^1(C_1^+)} \leq C|m|\Big(\|g_0\|_{H^1(\Omega)^N} + \|g_1\|_{H^2(\Omega)} + \|g_2\|_{H^{3/2}(\partial\Omega)^N}\Big)$$

for $k = 1, \ldots, N-1$. Taking $m \to 0$, this implies $(\partial_k \tilde{z}, \partial_k \tilde{h}) \in H^2(C_1^+)^N \times H^1(C_1^+)$ and

$$\|\partial_k \tilde{z}\|_{H^2(C_1^+)} + \|\partial_k \tilde{h}\|_{H^1(C_1^+)} \leq C(\|g_0\|_{H^1(\Omega)^N} + \|g_1\|_{H^2(\Omega)} + \|g_2\|_{H^{3/2}(\partial\Omega)^N})$$

for $1 \leq k \leq N-1$. Now, we will prove that $\Big(\frac{\partial \tilde{z}_i}{\partial \xi_N}, \frac{\partial \tilde{h}}{\partial \xi_N}\Big) \in H^2(C_1^+) \times H^1(C_1^+)$ for every $i = 1, \ldots, N$.

From (2.14) we have

$$-\frac{\partial^2 \tilde{z}_i}{\partial \xi_N^2} \sum_{k=1}^N \Big|\frac{\partial \xi_N}{\partial x_k}\Big|^2 + \frac{\partial \tilde{h}}{\partial \xi_N}\frac{\partial \xi_N}{\partial x_i} \in H^1(C_1^+), \quad \forall i = 1, \ldots N. \tag{2.17}$$

Then

$$-\Bigg(\sum_{k=1}^N \Big|\frac{\partial \xi_N}{\partial x_k}\Big|^2\Bigg)\Bigg(\sum_{i=1}^N \frac{\partial^3 \tilde{z}_i}{\partial \xi_N^3}\frac{\partial \xi_N}{\partial x_i}\Bigg) + \frac{\partial^2 \tilde{h}}{\partial \xi_N^2}\sum_{i=1}^N \Big|\frac{\partial \xi_N}{\partial x_i}\Big|^2 \in L^2(C_1^+). \tag{2.18}$$

On the other hand, from the divergence free condition (see (2.14)) we get

$$\sum_{i=1}^N \frac{\partial \tilde{z}_i}{\partial \xi_N}\frac{\partial \xi_N}{\partial x_i} = -\sum_{i=1}^N \Bigg(\sum_{k=1}^{N-1} \frac{\partial \tilde{z}_i}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_i}\Bigg) + \tilde{g_1}^* \in H^2(C_1^+),$$

so that

$$\sum_{i=1}^N \frac{\partial^3 \tilde{z}_i}{\partial \xi_N^3}\frac{\partial \xi_N}{\partial x_i} \in L^2(C_1^+). \tag{2.19}$$

From (2.18) and (2.19), we obtain that

$$\frac{\partial^2 \tilde{h}}{\partial \xi_N^2}\sum_{i=1}^N \Big|\frac{\partial \xi_N}{\partial x_i}\Big|^2 \in L^2(C_1^+)$$

and therefore $\tilde{h} \in H^2(C_1^+)$. Coming back to (2.17) we obtain that

$$\frac{\partial^3 \tilde{z}_i}{\partial \xi_N^3}\sum_{k=1}^N \Big|\frac{\partial \xi_N}{\partial x_k}\Big|^2 \in L^2(C_1^+), \quad \forall i = 1, \ldots, N.$$

Therefore $\tilde{h} \in H^2(C_1^+)$ and $\tilde{z} \in H^3(C_1^+)^N$, so that $(\partial_k z, \partial_k h) \in H^2(\Omega \cap \tilde{U})^N \times H^1(\Omega \cap \tilde{U})$ for $k = 1, \ldots, N$ and we can conclude that $(z, h) \in H^3(\Omega \cap \tilde{U})^N \times H^2(\Omega \cap \tilde{U})$ for every $\tilde{U} \subset\subset U$ with the estimate

$$\|z\|_{H^3(\Omega\cap\tilde{U})^N} + \|h\|_{H^2(\Omega\cap\tilde{U})} \leq C\Big(\|g_0\|_{H^1(\Omega\cap\tilde{U})^N} + \|g_1\|_{H^2(\Omega\cap\tilde{U})} + \|g_2\|_{H^{3/2}(\partial\Omega\cap\tilde{U})^N}\Big). \tag{2.20}$$

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This, together with (2.10), gives the following estimate for the solution of the stationary system (2.7):

$$\begin{aligned} &\|u\|_{H^3(\Omega)^N} + \|\theta\|_{H^2(\Omega)} \\ &\le C\Big(\|f_0\|_{H^1(\Omega)^N} + \|u_t\|_{H^1(\Omega)^N} + \|f_2\|_{H^{3/2}(\partial\Omega)^N} + \|Au\|_{H^{3/2}(\partial\Omega)^N}\Big). \end{aligned} \tag{2.21}$$

Now, to estimate the term $\|u_t(t)\|_{H^1(\Omega)^N}$ we multiply (2.4) by

$$\partial_t(B(u,\theta)) := -\nabla\cdot Du_t + \nabla\theta_t$$

and integrate in Ω. We get

$$-\int_\Omega u_t\nabla\cdot Du_t dx + \int_\Omega u_t\cdot\nabla\theta_t dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |B(u,\theta)|^2dx = \int_\Omega f_0\cdot\nabla\theta_t dx - \int_\Omega f_0\nabla\cdot Du_t dx.$$

Integrating by parts and using that f_0 belongs to W, we obtain

$$\begin{aligned} &\int_\Omega |\nabla u_t|^2dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |B(u,\theta)|^2dx - \int_{\partial\Omega} u_t\cdot(Du_t\cdot n)_{tg}d\sigma \\ &= \int_\Omega \nabla f_0\cdot\nabla u_t dx - \int_{\partial\Omega} f_0\cdot(Du_t\cdot n)_{tg}d\sigma. \end{aligned}$$

We use now $(Du_t\cdot n)_{tg} = \partial_t f_2 - \partial_t(Au)$:

$$\begin{aligned} &\int_\Omega |\nabla u_t|^2dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |B(u,\theta)|^2dx + \int_{\partial\Omega}\partial_t(Au)\cdot u_t d\sigma \\ &= \int_\Omega \nabla f_0\cdot\nabla u_t dx + \int_{\partial\Omega}(\partial_t f_2)\cdot u_t d\sigma + \int_{\partial\Omega}\partial_t(Au)\cdot f_0 d\sigma - \int_{\partial\Omega}\partial_t f_2\cdot f_0 d\sigma, \end{aligned}$$

for almost every $t\in(0,T)$. Coming back to (2.21), we get

$$\begin{aligned} &\|\nabla u_t\|^2_{L^2(\Omega)^N} + \|u\|^2_{H^3(\Omega)^N} + \frac{1}{2}\frac{d}{dt}\int_\Omega |B(u,\theta)|^2dx + \|\theta\|^2_{H^2(\Omega)} \\ &\le C\Big(\|f_0\|^2_{H^1(\Omega)^N} + \|f_2\|^2_{H^{3/2}(\partial\Omega)^N} + \|Au\|^2_{H^{3/2}(\partial\Omega)^N} + \int_{\partial\Omega}|\partial_t(Au)||u_t|d\sigma \\ &\quad + \int_{\partial\Omega}|\partial_t(Au)||f_0|d\sigma + \int_{\partial\Omega}|\partial_t f_2\cdot u_t|d\sigma + \int_{\partial\Omega}|\partial_t f_2\cdot f_0|d\sigma + \|u_t\|^2_{L^2(\Omega)^N}\Big), \end{aligned} \tag{2.22}$$

for almost every $t\in(0,T)$.

In order to estimate the third term in (2.22) we use that

$$H^{3/2}(\partial\Omega)\cdot H^{3/2}(\partial\Omega)\subset H^{3/2}(\partial\Omega) \quad \text{continuously.}$$

Then

$$\|Au\|^2_{H^{3/2}(\partial\Omega)^N} \le C\|A\|^2_{H^{3/2}(\partial\Omega)^{N\times N}}\|u\|^2_{H^{3/2}(\partial\Omega)^N} \le C\|A\|^2_{H^{3/2}(\partial\Omega)^{N\times N}}\|u\|^2_{H^2(\Omega)^N}.$$

From this estimate and (2.22) we obtain

$$\begin{aligned} &\|\nabla u_t\|^2_{L^2(Q)^N} + \|u\|^2_{L^2(H^3(\Omega)^N)} + \|B(u,\theta)\|^2_{L^\infty(L^2(\Omega)^N)} + \|\theta\|^2_{L^2(H^2(\Omega))} \\ &\leq C\Big(\|f_0\|^2_{L^2(H^1(\Omega)^N)} + \|f_2\|^2_{L^2(H^{3/2}(\partial\Omega)^N)} + \|A\|^2_{L^\infty(H^{3/2}(\partial\Omega)^{N\times N})}\|u\|^2_{L^2(H^2(\Omega)^N)} \\ &+ \iint_\Sigma (|\partial_t(Au)| + |\partial_t f_2|)(|u_t| + |f_0|)d\sigma dt + \|B(u_0,\theta(0))\|^2_{L^2(\Omega)^N} + \|u_t\|^2_{L^2(Q)^N}\Big), \end{aligned} \tag{2.23}$$

where $\theta(0)$ is defined (up to a constant) by

$$\begin{cases} -\Delta\theta(0)(\cdot) = -\nabla f_0(\cdot,0) & \text{in } \Omega, \\ \dfrac{\partial\theta(0)}{\partial n}(\cdot) = \Delta u_0(\cdot)\cdot n + f_0(\cdot,0)\cdot n & \text{on } \partial\Omega. \end{cases} \tag{2.24}$$

Now, we estimate the boundary terms in (2.23). First, we find

$$\begin{aligned} \iint_\Sigma |\partial_t(Au)|(|u_t| + |f_0|)d\sigma dt \leq C_\delta(\|A\|^4_{L^\infty(\Sigma)^{N\times N}}\|u_t\|^2_{L^2(Q)^N} + \|A_t\|^2_{L^2(\Sigma)^{N\times N}}\|u\|^2_{L^\infty(H^1(\Omega)^N)}) \\ + \delta(\|u_t\|^2_{L^2(H^1(\Omega)^N)} + \|f_0\|^2_{L^2(H^1(\Omega)^N)}) \end{aligned}$$

for any $\delta > 0$. The second term can be estimated as follows:

$$\iint_\Sigma |\partial_t f_2|(|u_t| + |f_0|)d\sigma dt \leq C_\delta\|\partial_t f_2\|^2_{L^2(H^{1/2}(\partial\Omega)^N)} + \delta(\|u_t\|^2_{L^2(H^1(\Omega)^N)} + \|f_0\|^2_{L^2(H^1(\Omega)^N)}).$$

Putting together these estimates and (2.23) we can deduce

$$\begin{aligned} &\|u_t\|^2_{L^2(H^1(\Omega)^N)} + \|u\|^2_{L^2(H^3(\Omega)^N)} + \|B(u,\theta)\|^2_{L^\infty(L^2(\Omega)^N)} + \|\theta\|^2_{L^2(H^2(\Omega))} \\ &\leq C\Bigg(\|f_0\|^2_{L^2(H^1(\Omega)^N)} + \|f_2\|^2_{L^2(H^{3/2}(\partial\Omega)^N)} + \|\partial_t f_2\|^2_{L^2(H^{1/2}(\partial\Omega)^N)} + \|B(u_0,\theta(0))\|^2_{L^2(\Omega)^N} \\ &\quad + \Big(1 + \|A\|^2_{L^\infty(H^{3/2}(\partial\Omega)^{N\times N})} + \|\partial_t A\|^2_{L^2(\Sigma)^{N\times N}} + \|A\|^4_{L^\infty(\Sigma)^{N\times N}}\Big)\|u\|^2_{Y_1}\Bigg). \end{aligned}$$

Using (2.2) in order to estimate $\|u\|^2_{Y_1}$ and elliptic estimates (2.24), we get

$$\begin{aligned} &\|u_t\|^2_{L^2(H^1(\Omega)^N)} + \|u\|^2_{L^2(H^3(\Omega)^N)} + \|B(u,\theta)\|^2_{L^\infty(L^2(\Omega)^N)} + \|\theta\|^2_{L^2(H^2(\Omega))} \\ &\leq C(A)\Big(\|f_0\|^2_{Y_1} + \|f_2\|^2_{L^2(H^{3/2}(\partial\Omega)^N)} + \|\partial_t f_2\|^2_{L^2(H^{1/2}(\partial\Omega)^N)} + \|u_0\|^2_{H^3(\Omega)^N}\Big), \end{aligned} \tag{2.25}$$

where

$$C(A) := Ce^{CT\|A\|^2_{P^0_\varepsilon}}(1 + \|A\|^4_{P^0_\varepsilon})\Big(1 + \|A\|^2_{L^\infty(H^{3/2}(\partial\Omega)^{N\times N})} + \|\partial_t A\|^2_{L^2(\Sigma)^{N\times N}} + \|A\|^4_{L^\infty(\Sigma)^{N\times N}}\Big).$$

Step 2. Taking into account the previous step, we will prove that the weak solution (u,θ) of (2.8) belongs to $H^4(\Omega)^N \times H^3(\Omega)$ whenever

$$g_0^* \in H^2(\Omega \cap U_0)^N, \quad g_1^* \in H^3(\Omega \cap U_0), \quad g_2^* \in H^{5/2}(\partial\Omega \cap U_0)^N, \tag{2.26}$$

also, ψ is a $W^{4,\infty}$ diffeomorphism. Here, we define

$$\tilde{X}_{1,3} := \{\tilde{z} \in H^3(C_1^+)^N : \tilde{z} = 0 \text{ on } \partial C_1^+ \cap \mathbb{R}_+^N,\ \tilde{z} \cdot \tilde{n} = 0 \text{ on } \partial \mathbb{R}_+^N \cap C_1\}.$$

Let us prove that $\tilde{z}$ satisfies $\delta_m^k \tilde{z} \in \tilde{X}_{1,3}$, for $k = 1, \ldots, N-1$ and $|m| < d/2$ (recall that $d = dist(\partial C_0^+, \partial C_1^+)$), where $\tilde{z}$ fulfills (2.14). We have the following estimates for G_0, G_1 and G_2 (which were defined right after (2.16)):

$$\|G_0\|_{H^1(C_1^+)^N} \le C|m| \Big(\|g_0^*\|_{H^2(\Omega)^N} + \|z\|_{H^3(\Omega)^N} + \|\nabla h\|_{H^1(\Omega)} \Big).$$

$$\|G_1\|_{H^2(C_1^+)} \le C|m| \Big(\|g_1^*\|_{H^3(\Omega)} + \|z\|_{H^3(\Omega)^N} \Big)$$

and

$$\|G_2\|_{H^{3/2}(\partial \mathbb{R}_+^N \cap C_1)^N} \le C|m| (\|\tilde{g}_2^*\|_{H^{5/2}(\partial \mathbb{R}_+^N \cap C_1)^N} + \|\tilde{z}\|_{H^{5/2}(\partial \mathbb{R}_+^N \cap C_1)^N}).$$

Then, using (2.26) together with the definition of g_i^* $(i = 0, 1, 2)$ given in (2.13) and the estimate (2.25) for the solutions of the stationary problems (2.8) and (2.12), we obtain

$$\|G_0\|_{H^1(C_1^+)^N} \le C|m| \Big(\|g_0\|_{H^2(\Omega)^N} + \|g_1\|_{H^2(\Omega)} + \|g_2\|_{H^{3/2}(\partial\Omega)^N} \Big),$$

$$\|G_1\|_{H^2(C_1^+)} \le C|m| \Big(\|g_0\|_{H^2(\Omega)^N} + \|g_1\|_{H^3(\Omega)} + \|g_2\|_{H^{3/2}(\partial\Omega)^N} \Big)$$

and

$$\|G_2\|_{H^{3/2}(\partial \mathbb{R}_+^N \cap C_1)^N} \le C|m| \Big(\|g_0\|_{H^1(\Omega)^N} + \|g_1\|_{H^2(\Omega)} + \|g_2\|_{H^{5/2}(\partial\Omega)^N} \Big).$$

In consequence, $(\delta_m^k \tilde{z}, \delta_m^k \tilde{h}) \in \tilde{X}_{1,3} \times H^2(C_1^+)$ and

$$\|\delta_m^k \tilde{z}\|_{H^3(C_1^+)^N} + \|\delta_m^k \tilde{h}\|_{H^2(C_1^+)} \le C|m| \Big(\|g_0\|_{H^2(\Omega)^N} + \|g_1\|_{H^3(\Omega)} + \|g_2\|_{H^{5/2}(\partial\Omega)^N} \Big)$$

for $k = 1, \ldots, N - 1$.

Arguing now as in **Step 1**, we find

$$\|u\|_{H^4(\Omega)^N} + \|h\|_{H^3(\Omega)} \le C \Big(\|g_0\|_{H^2(\Omega)^N} + \|g_1\|_{H^3(\Omega)} + \|g_2\|_{H^{5/2}(\partial\Omega)^N} \Big). \tag{2.27}$$

From (2.27) we obtain the estimate for the solution of the stationary system (2.7):

$$\|u\|_{H^4(\Omega)^N} + \|\theta\|_{H^3(\Omega)} \le C \Big(\|f_0\|_{H^2(\Omega)^N} + \|u_t\|_{H^2(\Omega)^N} + \|f_2\|_{H^{5/2}(\partial\Omega)^N} + \|Au\|_{H^{5/2}(\partial\Omega)^N} \Big), \tag{2.28}$$

for almost every $t \in (0, T)$. Now, in order to estimate the second term of the right-hand side of (2.28), we consider the system satisfied by $(\partial_t u, \partial_t \theta)$ (see (2.4)):

$$\begin{cases} \partial_t(u_t) - \nabla \cdot (Du_t) + \nabla \theta_t = \partial_t f_0 & \text{in} \quad Q, \\ \nabla \cdot u_t = 0 & \text{in} \quad Q, \\ u_t \cdot n = 0,\ (\sigma(u_t, \theta_t) \cdot n)_{tg} + (Au_t)_{tg} = \partial_t f_2 - (A_t u)_{tg} & \text{on} \quad \Sigma, \\ u_t(\cdot, 0) = \nabla \cdot Du_0(\cdot) - \nabla \theta(\cdot, 0) + f_0(\cdot, 0) & \text{in} \quad \Omega. \end{cases} \tag{2.29}$$

In virtue of Lemma 2.1 we have that (u_t, θ_t) is the strong solution of (2.29). Furthermore, we get $u_t \in Y_1$ and

$$\|u_t\|_{Y_1} \le e^{CT\|A\|^2_{P^0_\varepsilon}}(1+\|A\|^2_{P^0_\varepsilon})\Big(\|\partial_t f_0\|_{L^2(Q)^N} + \|f_0\|_{L^\infty(H^1(\Omega)^N)} + \|\partial_t f_2\|_{L^2(H^{1/2}(\partial\Omega)^N)} + \|\partial_t f_2\|_{H^{1/4+\varepsilon}(H^{-\varepsilon}(\Omega)^N)} + \|A_t u\|_{H^{1/4+\varepsilon}(H^{-\varepsilon}(\Omega)^N)} + \|A_t u\|_{L^2(H^{1/2}(\partial\Omega)^N)} + \|u_0\|_{H^3(\Omega)^N\cap W}\Big). \tag{2.30}$$

Therefore, from (2.28) and (2.30) we obtain

$$\begin{aligned}&\|u_t\|_{Y_1} + \|u\|_{L^2(H^4(\Omega)^N)} + \|\theta\|_{L^2(H^3(\Omega))}\\ &\le e^{CT\|A\|^2_{P^0_\varepsilon}}(1+\|A\|^2_{P^0_\varepsilon})\Big(\|f_0\|_{L^2(H^2(\Omega)^N)} + \|\partial_t f_0\|_{L^2(Q)^N} + \|f_2\|_{L^2(H^{5/2}(\partial\Omega)^N)}\\ &\quad + \|\partial_t f_2\|_{L^2(H^{1/2}(\partial\Omega)^N)} + \|\partial_t f_2\|_{H^{1/4+\varepsilon}(H^{-\varepsilon}(\Omega)^N)} + \|A_t u\|_{H^{1/4+\varepsilon}(H^{-\varepsilon}(\Omega)^N)}\\ &\quad + \|A_t u\|_{L^2(H^{1/2}(\partial\Omega)^N)} + \|Au\|_{L^2(H^{5/2}(\partial\Omega)^N)} + \|u_0\|_{H^3(\Omega)^N\cap W}\Big).\end{aligned} \tag{2.31}$$

Finally, we estimate $\|A_t u\|_{L^2(H^{1/2}(\partial\Omega)^N)}$, $\|A_t u\|_{H^{1/4+\varepsilon}(H^{-\varepsilon}(\Omega)^N)}$ and $\|Au\|_{L^2(H^{5/2}(\partial\Omega)^N)}$ by:

$$\|A_t u\|_{L^2(H^{1/2}(\partial\Omega)^N)} \le C\|A_t\|_{L^2(H^{1/2}(\partial\Omega)^{N\times N})}\|u\|_{L^\infty(H^2(\Omega)^N)}$$

$$\|A_t u\|_{H^{1/4+\varepsilon}(H^{-\varepsilon}(\Omega)^N)} \le C\|A_t\|_{H^{1/4+\varepsilon}(L^2(\partial\Omega)^{N\times N})}\Big(\|u\|_{L^2(H^3(\Omega)^N)} + \|u\|_{H^1(H^1(\Omega)^N)}\Big)$$

and

$$\|Au\|_{L^2(H^{5/2}(\partial\Omega)^N)} \le C\Big(\|A\|_{L^\infty(H^{3/2}(\partial\Omega)^{N\times N})}\|u\|_{L^2(H^3(\Omega)^N)} + \|A\|_{L^2(H^{5/2}(\partial\Omega)^{N\times N})}\|u\|_{L^\infty(H^2(\Omega)^N)}\Big).$$

Using (2.25), (2.31) and the previous estimates, we find the desired estimate (2.5). This concludes the proof of Theorem 2.1. □

3. Carleman inequality for the adjoint system

In this section we will mainly prove a Carleman estimate for the adjoint system of (1.3). In order to do so, we are going to introduce some weight functions. Let ω_0 be a nonempty open subset of $\mathbb{R}^N$ such that $\omega_0 \Subset \omega_1 \Subset \omega' \Subset \omega$ and $\eta \in C^2(\overline{\Omega})$ such that

$$|\nabla\eta| > 0 \text{ in } \overline{\Omega}\setminus\omega_0, \quad \eta > 0 \text{ in } \Omega \quad \text{and} \quad \eta \equiv 0 \text{ on } \partial\Omega.$$

The existence of such a function η is proved in [15]. Then, for all $\lambda \ge 1$ we consider the following weight functions:

$$\begin{aligned}&\alpha(x,t) = \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x)}}{(t(T-t))^{11}}, \quad \xi(x,t) = \frac{e^{\lambda\eta(x)}}{(t(T-t))^{11}},\\ &\alpha^*(t) = \max_{x\in\overline{\Omega}}\alpha(x,t), \qquad \xi^*(t) = \min_{x\in\overline{\Omega}}\xi(x,t),\\ &\widehat{\alpha}(t) = \min_{x\in\overline{\Omega}}\alpha(x,t), \qquad \widehat{\xi}(t) = \max_{x\in\overline{\Omega}}\xi(x,t).\end{aligned} \tag{3.1}$$

We consider now a backwards nonhomogeneous system associated to the Stokes equation:

$$\begin{cases} -\varphi_t - \nabla\cdot(D\varphi) + \nabla\pi = g & \text{in} \quad Q, \\ \nabla\cdot\varphi = 0 & \text{in} \quad Q, \\ \varphi\cdot n = 0,\ (\sigma(\varphi,\pi)\cdot n)_{tg} + (A^t(x,t)\varphi)_{tg} = 0 & \text{on} \quad \Sigma, \\ \varphi(\cdot,T) = \varphi^T(\cdot) & \text{in} \quad \Omega, \end{cases} \tag{3.2}$$

where $g \in L^2(Q)^N$ and $\varphi^T \in H$. Our Carleman estimate is given in the following proposition.

Proposition 3.1. *Let $A \in P^1_\varepsilon \cap P^2$. There exists a constant λ_0, such that for any $\lambda > \lambda_0$ there exist two constants $C(\lambda) > 0$ increasing on $\|A\|_{P^1_\varepsilon \cap P^2}$ and $s_0(\lambda) > 0$ such that for any $i \in \{1,\dots,N\}$, any $g \in L^2(Q)^N$ and any $\varphi^T \in H$, the solution of (3.2) satisfies*

$$\begin{aligned} &s^3 \iint\limits_Q e^{-6s\alpha^*}(\xi^*)^3|\varphi|^2 dxdt \\ &\le C\Big(\iint\limits_Q e^{-4s\alpha^*}|g|^2 dxdt + s^7 \sum_{j=1, j\neq i}^N \int_0^T\int_{\omega'} e^{-4s\hat\alpha - 2s\alpha^*}(\hat\xi)^{12}|\varphi_j|^2 dxdt\Big) \end{aligned} \tag{3.3}$$

for every $s \ge s_0$.

Before giving the proof of Proposition 3.1, we present some technical results. We first present a Carleman inequality proved in [12] for a general heat equation with Fourier boundary conditions. To this end, let us introduce the system

$$\begin{cases} -\psi_t - \Delta\psi = f_1 + \nabla\cdot f_2 & \text{in} \quad Q, \\ (\nabla\psi + f_2)\cdot n = f_3 & \text{on} \quad \Sigma, \\ \psi(\cdot,T) = \psi^T(\cdot) & \text{in} \quad \Omega, \end{cases} \tag{3.4}$$

where $f_1 \in L^2(Q)$, $f_2 \in L^2(Q)^N$ and $f_3 \in L^2(\Sigma)$. We present now this result:

Lemma 3.1. *Under the previous assumptions on f_1, f_2 and f_3, there exist $\overline{\lambda}, \sigma_1, \sigma_2$ and C, only depending on Ω and ω, such that, for any $\lambda \ge \overline{\lambda}$, any $s \ge \overline{s} = \sigma_1(e^{\sigma_2\lambda}T + T^2)$ and any $\psi^T \in L^2(\Omega)$, the weak solution to (3.4) satisfies*

$$\begin{aligned} &\iint\limits_Q e^{-2s\alpha}(s\lambda^2\xi|\nabla\psi|^2 + s^3\lambda^4\xi^3|\psi|^2)dxdt + s^2\lambda^3\iint\limits_\Sigma e^{-2s\alpha}\xi^2|\psi|^2 d\sigma dt \\ &\le C\Big(\iint\limits_Q e^{-2s\alpha}(|f_1|^2 + s^2\lambda^2\xi^2|f_2|^2)dxdt \\ &\quad + s\lambda\iint\limits_\Sigma e^{-2s\alpha}\xi|f_3|^2 d\sigma dt + s^3\lambda^4 \iint\limits_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^3|\psi|^2 dxdt\Big). \end{aligned} \tag{3.5}$$

The next lemma is a result for elliptic equations with non homogeneous boundary condition that can be found in [18] (see also [11]).

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Lemma 3.2. *Let $y \in H^1(\Omega)$ satisfy*

$$\Delta y = f_0 + \sum_{j=1}^{N} \frac{\partial f_j}{\partial x_j}, \quad \text{in } \Omega\,; \quad y = g, \quad \text{on } \partial\Omega,$$

with $f_0, f_j \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$. Then there exist three constants $C > 0$, $\hat{\lambda} > 1$ and $\hat{\tau} > 1$ such that for any $\lambda \geq \hat{\lambda}$ and any $\tau \geq \hat{\tau}$, we have

$$\begin{aligned} &\int_\Omega |\nabla y|^2 e^{2\tau e^{\lambda\eta}} dx + \tau^2\lambda^2 \int_\Omega e^{2\lambda\eta}|y|^2 e^{2\tau e^{\lambda\eta}} dx \\ &\quad \leq C\Big(\tau^{1/2} e^{2\tau} \|g\|^2_{H^{1/2}(\partial\Omega)} + \tau^{-1}\lambda^{-2} \int_\Omega e^{-\lambda\eta}|f_0|^2 e^{2\tau e^{\lambda\eta}} dx \\ &\qquad + \sum_{j=0}^{N} \tau \int_\Omega e^{\lambda\eta}|f_j|^2 e^{2\tau e^{\lambda\eta}} dx + \int_{\omega_1} (|\nabla y|^2 + \tau^2\lambda^2 e^{2\lambda\eta}|y|^2) e^{2\tau e^{\lambda\eta}} dx\Big). \end{aligned} \tag{3.6}$$

Remark 3.1. We can eliminate the local integral of $|\nabla y|^2$ in (3.6) at the price of having a local term of $|y|^2$ in a open set ω_2 satisfying $\omega_1 \Subset \omega_2 \Subset \omega'$. For these details, we invite to the interested reader to see [11].

The next technical result corresponds to the Lemma 3 in [6].

Lemma 3.3. *Let $r \in \mathbb{R}$. There exists $C > 0$ depending only on r, Ω, ω_0 and η such that, for every $T > 0$ and every $u \in L^2(0, T; H^1(\Omega))$,*

$$s^2\lambda^2 \iint_Q e^{-2s\alpha}\xi^{r+2}|u|^2 dxdt \leq C\Big(\iint_Q e^{-2s\alpha}\xi^r|\nabla u|^2 dxdt + s^2\lambda^2 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^{r+2}|u|^2 dxdt\Big), \tag{3.7}$$

foe every $\lambda \geq C$ and every $s \geq CT^{22}$.

Remark 3.2. In [6], [12] and [18] slightly different weight functions are used to prove the above results. However, this does not change the result since the important property is that α goes to $+\infty$ when t tends to 0 and T.

We will now prove Proposition 3.1. Without any lack of generality, we treat the case $N = 2$ and $i = 2$. The arguments can be easily extended to the general case. Let us introduce (w, q) and (z, r), the solutions of the following systems:

$$\begin{cases} -w_t - \nabla\cdot(Dw) + \nabla q = \rho g & \text{in} \quad Q, \\ \nabla\cdot w = 0 & \text{in} \quad Q, \\ w\cdot n = 0,\ (\sigma(w,q)\cdot n)_{tg} + (A^t(x,t)w)_{tg} = 0 & \text{on} \quad \Sigma, \\ w(\cdot, T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{3.8}$$

and

$$\begin{cases} -z_t - \nabla\cdot(Dz) + \nabla r = -\rho'\varphi & \text{in} \quad Q, \\ \nabla\cdot z = 0 & \text{in} \quad Q, \\ z\cdot n = 0,\ (\sigma(z,r)\cdot n)_{tg} + (A^t(x,t)z)_{tg} = 0 & \text{on} \quad \Sigma, \\ z(\cdot, T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{3.9}$$

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where $\rho(t) = e^{-2s\alpha^*}$. Adding (3.8) and (3.9), we see that $(w + z, q + r)$ solves the same system as $(\rho\varphi, \rho\pi)$, where (φ, π) is the solution of (3.2). By uniqueness of the Stokes system with Navier-slip boundary conditions, we have

$$\rho\varphi = w + z \quad \text{and} \quad \rho\pi = q + r. \tag{3.10}$$

For system (3.8) we will use Lemma 2.1 and the regularity estimate (2.2), namely

$$\|w\|^2_{L^2(0,T;H^2(\Omega)^2)} + \|w\|^2_{H^1(0,T;L^2(\Omega)^2)} \le C\|\rho g\|^2_{L^2(Q)^2}, \tag{3.11}$$

and for the system (3.9) we will use the ideas of [4] and [6].

We apply the operator ∇ to the equation satisfied by z_1 and we denote $\psi := \nabla z_1$. Then ψ satisfies

$$-\psi_t - \Delta\psi = -\nabla(\rho'\varphi_1) - \nabla\partial_1 r \quad \text{in } Q.$$

Using Lemma 3.1 with $f_1 = -\nabla(\rho'\varphi_1) - \nabla\partial_1 r$ and $f_2 = 0$, we obtain

$$\begin{aligned} s^3 \iint\limits_Q e^{-2s\alpha}\xi^3|\psi|^2 dxdt \le C\Big(& s^3 \int_0^T\int\limits_{\omega_1} e^{-2s\alpha}\xi^3|\psi|^2 dxdt \\ &+ s \iint\limits_\Sigma e^{-2s\alpha}\xi\Big|\frac{\partial\nabla z_1}{\partial n}\Big|^2 d\sigma dt + \iint\limits_Q e^{-2s\alpha}|\nabla(\rho'\varphi_1) + \nabla\partial_1 r|^2 dxdt\Big) \end{aligned} \tag{3.12}$$

for every $\lambda \ge \overline{\lambda}$ and $s \ge \overline{s_0}$. Here and in the following, C will denote a generic constant depending on Ω, ω and λ.

The rest of the proof is divided in three steps.

a) In step 1, using Lemma 3.3 we estimate global integrals of z_1 and z_2. In addition, we partially estimate the pressure in the right-side of (3.12).
b) In step 2, we will estimate the normal derivative appearing in the right-hand side of (3.12) and the global term of the pressure obtained in step 1.
c) In step 3, we will estimate all the local terms by a local term of φ_1.

Step 1. *Estimate of* z_1. We use Lemma 3.3 with $u = \nabla z_1$ and $r = 3$:

$$s^5 \iint\limits_Q e^{-2s\alpha}\xi^5|z_1|^2 dxdt \le C\Big(s^3 \iint\limits_Q e^{-2s\alpha}\xi^3|\psi|^2 dxdt + s^5 \int_0^T\int\limits_{\omega_0} e^{-2s\alpha}\xi^5|z_1|^2 dxdt\Big) \tag{3.13}$$

for every $s \ge C$.

Estimate of z_2. Let us first establish a general estimate: $\forall\varepsilon' > 0, \exists C \in \mathbb{R}$:

$$\|u\|_{(H^{1/2+\varepsilon'}(\Omega)^2\cap H)'} \le C(\|u_1\|_{L^2(\Omega)} + \|u_1 n_1\|_{L^2(\partial\Omega)} + \|\partial_1 u_1\|_{H^{-1/2}(\Omega)}) \le C\|u_1\|_{H^{1/2+\varepsilon'}(\Omega)}, \ \forall u \in W. \tag{3.14}$$

Indeed, for any $f \in H_{\varepsilon'} := H^{1/2+\varepsilon'}(\Omega)^2 \cap H$, we have (after an integration by parts)

$$\int\limits_\Omega u \cdot f\, dx = \int\limits_\Omega u_1 f_1\, dx - \int\limits_{\partial\Omega} u_1 n_1 \tilde{f}_2\, d\sigma + \int\limits_\Omega \partial_1 u_1 \tilde{f}_2\, dx, \tag{3.15}$$

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where $\tilde{f}_2 \in H^{1/2+\varepsilon'}(\Omega)$ satisfies

$$\partial_2 \tilde{f} = f_2 \text{ a.e. } \Omega \quad \textit{and} \quad \|\tilde{f}_2\|_{H^{1/2+\varepsilon'}(\Omega)} \le C\|f_2\|_{H^{1/2+\varepsilon'}(\Omega)} \le C\|f\|_{H_{\varepsilon'}}.$$

Then, from (3.15), we readily obtain (3.14).

Let us now apply (3.14) for $u := z$. We deduce

$$\forall \varepsilon' > 0,\ \exists C \in \mathbb{R} : \|z\|_{(H_{\varepsilon'})'} \le C\|z_1\|_{H^{1/2+\varepsilon'}(\Omega)},$$

so that, using that $H^{1/2+\varepsilon'}(\Omega)$ is the interpolation space $(H^1(\Omega), L^2(\Omega))_{1/2+\varepsilon',2}$, we find

$$s^{4-2\varepsilon'} \int_0^T e^{-2s\alpha^*}(\xi^*)^{4-2\varepsilon'} \|z\|^2_{(H_{\varepsilon'})'} dt \le Cs^3 \iint_Q e^{-2s\alpha^*}(\xi^*)^3 \Big(s^2(\xi^*)^2|z_1|^2 + |\nabla z_1|^2\Big) dxdt. \tag{3.16}$$

Putting together (3.12), (3.13) and (3.16) we have for the moment

$$\begin{aligned} s^5 \iint_Q e^{-2s\alpha}\xi^5|z_1|^2 dxdt + s^{4-2\varepsilon'} \int_0^T e^{-2s\alpha^*}(\xi^*)^{4-2\varepsilon'} \|z\|^2_{(H_{\varepsilon'})'} dt + s^3 \iint_Q e^{-2s\alpha}\xi^3|\psi|^2 dxdt \\ \le C\Big(\int_0^T \int_{\omega_1} e^{-2s\alpha}(s^5\xi^5|z_1|^2 + s^3\xi^3|\nabla z_1|^2) dxdt \\ + s\iint_\Sigma e^{-2s\alpha}\xi\Big|\frac{\partial \nabla z_1}{\partial n}\Big|^2 d\sigma dt + \iint_Q e^{-2s\alpha}|\nabla(\rho'\varphi_1) + \nabla\partial_1 r|^2 dxdt\Big) \end{aligned} \tag{3.17}$$

for every $s \ge C$.

Taking into account that

$$|\alpha_t^*| \le C(\xi^*)^{12/11}, \quad |\rho'| \le Cs\rho(\xi^*)^{12/11} \tag{3.18}$$

and (3.10), we obtain

$$\begin{aligned} \iint_Q e^{-2s\alpha}|\nabla(\rho'\varphi_1)|^2 dxdt = \iint_Q e^{-2s\alpha}|\rho'|^2|\rho|^{-2}|\nabla(\rho\varphi_1)|^2 dxdt \\ \le C\Big(s^2 \iint_Q e^{-2s\alpha}(\xi^*)^{24/11}|\nabla w_1|^2 + s^2 \iint_Q e^{-2s\alpha}(\xi^*)^{24/11}|\nabla z_1|^2 dxdt\Big). \end{aligned} \tag{3.19}$$

The fact that $s^2 e^{-2s\alpha^*}(\xi^*)^{24/11}$ is bounded allows us to estimate the first term in the right-hand side of (3.19) using (3.11). On the other hand, the second term in the right-hand side of (3.19) can be absorbed by the third term in the left-hand side of (3.17).

Additionally, using the divergence-free condition on the equation of (3.9), we see that

$$\Delta r = 0 \quad \text{in } Q,$$

then

$$\Delta(\nabla\partial_1 r) = 0 \quad \text{in } Q.$$

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Using Lemma 3.2 with $y = \nabla\partial_1 r$ and Remark 3.1 we obtain

$$\tau^2 \int_\Omega e^{2\lambda\eta}|\nabla\partial_1 r|^2 e^{2\tau e^{\lambda\eta}}dx \le C\Big(\tau^{1/2}e^{2\tau}\|\nabla\partial_1 r\|^2_{H^{1/2}(\partial\Omega)} + \tau^2 \int_{w_2} e^{2\lambda\eta}|\nabla\partial_1 r|^2 e^{2\tau e^{\lambda\eta}}dx\Big)$$

for every $\tau \ge C$. Now we take

$$\tau = \frac{s}{(t(T-t))^{11}},$$

multiply the last inequality by

$$\exp\Big(-2s\frac{e^{2\lambda\|\eta\|_\infty}}{(t(T-t))^{11}}\Big),$$

and integrate with respect to t in $(0,T)$ to obtain

$$\iint_Q e^{-2s\alpha}|\nabla\partial_1 r|^2 dxdt \le C\Big(s^{-3/2}\int_0^T e^{-2s\alpha^*}(\xi^*)^{-3/2}\|\nabla\partial_1 r\|^2_{H^{1/2}(\partial\Omega)}dt + \int_0^T\int_{\omega_2} e^{-2s\alpha}\xi^2|\nabla\partial_1 r|^2 dxdt\Big),$$

for all $s \ge C$.

Combining this with (3.17) and (3.19), we have for the moment

$$\begin{aligned} &s^5\iint_Q e^{-2s\alpha}\xi^5|z_1|^2dxdt + s^{4-2\varepsilon'}\int_0^T e^{-2s\alpha^*}(\xi^*)^{4-2\varepsilon'}\|z\|^2_{(H_{\varepsilon'})'}dt + s^3\iint_Q e^{-2s\alpha}\xi^3|\nabla z_1|^2dxdt \\ &\le C\Big(\|\rho g\|^2_{L^2(Q)^2} + s\iint_\Sigma e^{-2s\alpha}\xi\Big|\frac{\partial\nabla z_1}{\partial n}\Big|^2 d\sigma dt + s^{-3/2}\int_0^T e^{-2s\alpha^*}(\xi^*)^{-3/2}\|\nabla\partial_1 r\|^2_{H^{1/2}(\partial\Omega)}dt \\ &\quad + \int_0^T\int_{\omega_2} e^{-2s\alpha}\xi^2|\nabla\partial_1 r|^2dxdt + \int_0^T\int_{\omega_1} e^{-2s\alpha}(s^5\xi^5|z_1|^2 + s^3\xi^3|\nabla z_1|^2)dxdt\Big), \end{aligned} \tag{3.20}$$

for every $s \ge C$.

Step 2. In this step we deal with the boundary terms in (3.20), i.e.,

$$s\iint_\Sigma e^{-2s\alpha}\xi\Big|\frac{\partial\nabla z_1}{\partial n}\Big|^2 d\sigma dt \quad\text{and}\quad s^{-3/2}\int_0^T e^{-2s\alpha^*}(\xi^*)^{-3/2}\|\nabla\partial_1 r\|^2_{H^{1/2}(\partial\Omega)}dt.$$

Let us start by defining

$$\check z := \check\theta(t)z, \quad \check r := \check\theta(t)r, \quad \check\theta(t) := s^{1-\varepsilon'}e^{-s\alpha^*}(\xi^*)^{10/11-\varepsilon'}.$$

From (3.9), we see that $(\check z, \check r)$ is the solution of the Stokes system:

$$\begin{cases} -\check z_t - \nabla\cdot(D\check z) + \nabla\check r = -(\check\theta)'z - \check\theta\rho'\varphi & \text{in} \quad Q, \\ \nabla\cdot\check z = 0 & \text{in} \quad Q, \\ \check z\cdot n = 0,\ (\sigma(\check z,\check r)\cdot n)_{tg} + (A^t(x,t)\check z)_{tg} = 0 & \text{on} \quad \Sigma, \\ \check z(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{3.21}$$

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For this system, we have

$$\begin{aligned}\|\check{z}\|^2_{L^2(0,T;H^{3/2-\varepsilon'}(\Omega)^2)} &\le C\Big(\|s^{2-\varepsilon'}e^{-s\alpha^*}(\xi^*)^{2-\varepsilon'}z\|^2_{L^2(0,T;(H_{\varepsilon'})')} + \|s^2e^{-s\alpha^*}(\xi^*)^2\rho\varphi\|^2_{L^2(Q)^2}\Big)\\ &\le C\Big(\|s^{2-\varepsilon'}e^{-s\alpha^*}(\xi^*)^{2-\varepsilon'}z\|^2_{L^2(0,T;(H_{\varepsilon'})')} + \|s^2e^{-s\alpha^*}(\xi^*)^2 w\|^2_{L^2(Q)^2}\Big).\end{aligned} \tag{3.22}$$

Observe that this inequality comes from Lemma 2.1 with a right-hand side in the interpolation space ([20])

$$(L^2(0,T;W'),L^2(Q))_{1/2+\varepsilon',2} = L^2(0,T;(H_{\varepsilon'})').$$

The fact that $s^{3/2}e^{-s\alpha^*}(\xi^*)^{3/2}$ is bounded allows us to use (3.11) and conclude that $\|\check{z}\|^2_{L^2(0,T;H^{3/2-\varepsilon'}(\Omega)^2)}$ is bounded by the left-hand side of (3.20) and $\|\rho g\|^2_{L^2(Q)^2}$. Using that

$$L^2(\Omega)^2 = ((H_{\varepsilon'})', H^{3/2-\varepsilon'}(\Omega)^2)_{3/4-\varepsilon'/2,2},$$

we deduce that $s^{7/2-3\varepsilon'}\|e^{-s\alpha^*}(\xi^*)^{7/4-3\varepsilon'/2}z\|^2_{L^2(Q)^2}$ is bounded by the left-hand side of (3.20) and $\|\rho g\|^2_{L^2(Q)^2}$. Taking $\varepsilon' > 0$ small enough, we deduce in particular that

$$s^3 \iint\limits_Q e^{-2s\alpha^*}(\xi^*)^3|z|^2 dx\,dt$$

is bounded by the left-hand side of (3.20) and $\|\rho g\|^2_{L^2(Q)^2}$.

Next, we define

$$z^* := \theta^*(t)z, \quad r^* := \theta^*(t)r, \quad \theta^*(t) := s^{1/2}e^{-s\alpha^*}(\xi^*)^{9/22}.$$

From (3.9), we see that (z^*, r^*) is the solution of (3.21) with $\check{\theta}$ replaced by θ^*. Using again (2.2) and taking into account (3.18), we deduce

$$\begin{aligned}&\|z^*\|^2{}_{L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)} + \|r^*\|^2_{L^2(0,T;H^1(\Omega))} + \|\theta^* z_t\|^2_{L^2(0,T;L^2(\Omega)^2)}\\ &\quad\le C\Big(\|s^{3/2}e^{-s\alpha^*}(\xi^*)^{3/2}z\|^2_{L^2(Q)^2} + \|s^{3/2}e^{-s\alpha^*}(\xi^*)^{3/2}w\|^2_{L^2(Q)^2}\Big).\end{aligned} \tag{3.23}$$

Arguing as before, we conclude that $\|z^*\|^2_{L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)}$ is bounded by the left-hand side of (3.20) and $\|\rho g\|^2_{L^2(Q)^2}$.

Now, let

$$\hat{z} := \hat{\theta}(t)z, \quad \hat{r} := \hat{\theta}(t)r, \quad \hat{\theta} := s^{-1/2}e^{-s\alpha^*}(\xi^*)^{-15/22}.$$

From (3.9), $(\hat{z}, \hat{r})$ is the solution of (3.21) with $\check{\theta}$ replaced by $\hat{\theta}$. Observe that the right-hand side of this system can be considered in $L^2(0,T;H^2(\Omega)^2) \cap H^1(0,T;L^2(\Omega)^2)$ and thus, using the regularity estimate (2.5) we have

$$\begin{aligned}&\|\hat{z}\|^2_{L^2(0,T;H^4(\Omega)^2)\cap H^1(0,T;H^2(\Omega)^2)\cap H^2(0,T;L^2(\Omega)^2)} + \|\hat{r}\|^2_{L^2(0,T;H^3(\Omega))}\\ &\le C\Big(\|\hat{\theta}' z\|^2_{L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)} + \|\hat{\theta}\rho'\varphi\|^2_{L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)}\Big)\\ &\le C\Big(\|\rho g\|^2_{L^2(Q)^2} + \|z^*\|^2_{L^2(0,T;H^2(\Omega)^2)} + \|\theta^* z_t\|^2_{L^2(0,T;L^2(\Omega)^2)} + \|s^{3/2}e^{-s\alpha^*}(\xi^*)^{3/2}z\|^2_{L^2(Q)^2}\Big).\end{aligned} \tag{3.24}$$

From (3.23), the right-hand side of (3.24) is bounded by

$$\|s^{3/2}e^{-s\alpha^*}(\xi^*)^{3/2}z\|^2_{L^2(Q)^2} \quad \text{and} \quad \|\rho g\|^2_{L^2(Q)^2}.$$

Coming back to (3.20), we find in particular

$$\begin{aligned}
&s^5\iint\limits_Q e^{-2s\alpha}\xi^5|z_1|^2dxdt + s^3\iint\limits_Q e^{-2s\alpha^*}(\xi^*)^3|z_2|^2dxdt + \|z^*\|^2_{L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)}\\
&\quad + \|\hat{z}\|^2_{L^2(0,T;H^4(\Omega)^2)\cap H^2(0,T;L^2(\Omega)^2)} + \|\hat{r}\|^2_{L^2(0,T;H^3(\Omega))}\\
&\le C\Big(\|\rho g\|^2_{L^2(Q)^2} + s\iint\limits_\Sigma e^{-2s\alpha}\xi\Big|\frac{\partial\nabla z_1}{\partial n}\Big|^2d\sigma dt + s^{-3/2}\int_0^T e^{-2s\alpha^*}(\xi^*)^{-3/2}\|\nabla\partial_1 r\|^2_{H^{1/2}(\partial\Omega)}dt\\
&\qquad + \int_0^T\int_{\omega_2} e^{-2s\alpha}\xi^2|\nabla\partial_1 r|^2dxdt + \int_0^T\int_{\omega_1} e^{-2s\alpha}(s^5\xi^5|z_1|^2 + s^3\xi^3|\nabla z_1|^2)dxdt\Big).
\end{aligned} \tag{3.25}$$

Observe that the boundary term

$$s^{-3/2}\int_0^T e^{-2s\alpha^*}(\xi^*)^{-3/2}\|\nabla\partial_1 r\|^2_{H^{1/2}(\partial\Omega)}dt$$

can be absorbed by the fifth term of the left-hand side of (3.25).

In order to estimate the other boundary term, we notice that α and ξ coincide with α^* and ξ^* respectively on Σ, so that

$$s\iint\limits_\Sigma e^{-2s\alpha}\xi\Big|\frac{\partial\nabla z_1}{\partial n}\Big|^2d\sigma dt = s\iint\limits_\Sigma e^{-2s\alpha^*}\xi^*\Big|\frac{\partial\nabla z_1}{\partial n}\Big|^2d\sigma dt \le Cs\int_0^T e^{-2s\alpha^*}\xi^*\|z_1\|^2_{H^{5/2+\varepsilon}(\Omega)}dt \tag{3.26}$$

for every $\varepsilon > 0$. Taking $\varepsilon = \dfrac{1}{70}$ (any $0 < \varepsilon < \frac{1}{70}$ would work) and thanks to an interpolation argument between the spaces $L^2(L^2)$ and $L^2(H^4)$, we obtain

$$\begin{aligned}
&s^{43/35}\int_0^T e^{-2s\alpha^*}\xi^*\|z_1\|^2_{H^{88/35}(\Omega)}dt\\
&\qquad \le C\Big(s^5\int_0^T e^{-2s\alpha^*}(\xi^*)^5\|z_1\|^2_{L^2(\Omega)}dt + s^{-1}\int_0^T e^{-2s\alpha^*}(\xi^*)^{-15/11}\|z_1\|^2_{H^4(\Omega)}dt\Big),
\end{aligned}$$

for every $s \ge C$. Coming back to (3.26) and using the above inequality, the boundary term

$$s\iint\limits_\Sigma e^{-2s\alpha}\xi\Big|\frac{\partial\nabla z_1}{\partial n}\Big|^2d\sigma dt$$

can be absorbed by the left-hand side of (3.25). This ends Step 2.

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Thus, at this point we have

$$\begin{aligned}
&s^5 \iint\limits_Q e^{-2s\alpha}\xi^5|z_1|^2dxdt + s^3 \iint\limits_Q e^{-2s\alpha^*}(\xi^*)^3|z_2|^2dxdt \\
&\quad + \|\hat{\theta}z\|^2_{L^2(0,T;H^4(\Omega)^2)\cap H^2(0,T;L^2(\Omega)^2)} + \|\theta^* z\|^2_{L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)} \\
&\quad \le C\Big(\|\rho g\|^2_{L^2(Q)^2} + \int_0^T\int_{\omega_2} e^{-2s\alpha}\xi^2|\nabla\partial_1 r|^2dxdt + \int_0^T\int_{\omega_1} e^{-2s\alpha}(s^5\xi^5|z_1|^2 + s^3\xi^3|\nabla z_1|^2)dxdt\Big)
\end{aligned} \tag{3.27}$$

for every $s \ge C$.

Step 3. In this step, we estimate the local term on $\nabla\partial_1 r$ in the right-hand side of (3.27). The other two local terms can be estimated in an easier way.

Let ω_3 be a open subset satisfying $\omega_2 \Subset \omega_3 \Subset \omega'$ and let $\rho_1 \in C^2_c(\omega_3)$ with $\rho_1 \equiv 1$ in ω_2 and $\rho_1 \ge 0$. Then, integrating by parts and using that $\Delta r = 0$ we get

$$\int_0^T\int_{\omega_2} e^{-2s\alpha}\xi^2|\nabla\partial_1 r|^2dxdt \le C\int_0^T\int_{\omega_3} \Delta(\rho_1 e^{-2s\alpha}\xi^2)|\partial_1 r|^2dxdt.$$

From (3.9) and the estimate

$$|\Delta(\rho_1 e^{-2s\alpha}\xi^2)| \le Cs^2e^{-2s\alpha}\xi^4 1_{\omega_3}, \quad s \ge C,$$

we obtain

$$\int_0^T\int_{\omega_2} e^{-2s\alpha}\xi^2|\nabla\partial_1 r|^2dxdt \le Cs^2\int_0^T\int_{\omega_3} e^{-2s\alpha}\xi^4(|z_{1,t}|^2 + |\Delta z_1|^2 + |\rho'\varphi_1|^2)dxdt \tag{3.28}$$

for every $s \ge C$. We will now estimate the two first terms in the last integral of (3.28), the third one being estimated in an easier way.

i) Estimate of $z_{1,t}$. We integrate by parts with respect to t:

$$\begin{aligned}
s^2\int_0^T\int_{\omega_3} e^{-2s\alpha}\xi^4|z_{1,t}|^2dxdt &= \frac{s^2}{2}\int_0^T\int_{\omega_3} \partial_{tt}(e^{-2s\alpha}\xi^4)|z_1|^2dxdt - s^2\int_0^T\int_{\omega_3} e^{-2s\alpha}\xi^4 z_{1,tt}z_1dxdt \\
&\le C\Big(s^4\int_0^T\int_{\omega_3} e^{-2s\alpha}(\xi)^{68/11}|z_1|^2dxdt + s^2\int_0^T\int_{\omega_3} \hat{\theta}|z_{1,tt}|\hat{\theta}^{-1}e^{-2s\alpha}\xi^4|z_1|dxdt\Big),
\end{aligned}$$

where we recall that $\hat{\theta} := s^{-1/2}e^{-s\alpha^*}(\xi^*)^{-15/22}$.

Using Young's inequality for the second term we obtain for every $\varepsilon > 0$

$$\begin{aligned}
&s^2\int_0^T\int_{\omega_3} e^{-2s\alpha}\xi^4|z_{1,t}|^2dxdt \\
&\le C\Big(s^4\int_0^T\int_{\omega_3} e^{-2s\alpha}\xi^7|z_1|^2dxdt + \varepsilon\int_0^T\int_{\omega_3} |\hat{\theta}|^2|z_{1,tt}|^2dxdt + C(\varepsilon)s^5\int_0^T\int_{\omega_3} e^{-4s\alpha+2s\alpha^*}\xi^{10}|z_1|^2dxdt\Big).
\end{aligned} \tag{3.29}$$

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The second term in the right-hand side of the above inequality can be absorbed by the left-hand side of (3.27).

ii) Estimate of Δz_1. Let w_4 be an open subset such that $w_3 \Subset w_4 \Subset \omega'$ and let $\rho_2 \in C_c^2(w_4)$ with $\rho_2 \equiv 1$ in ω_3 and $\rho_2 \geq 0$. Then, an integration by parts gives

$$s^2 \int_0^T \int_{\omega_3} e^{-2s\alpha}\xi^4 |\Delta z_1|^2 dxdt \leq s^2 \int_0^T \int_{\omega_4} \rho_2^2 e^{-2s\alpha}\xi^4 |\Delta z_1|^2 dxdt$$
$$= - s^2 \int_0^T \int_{\omega_4} \nabla(\rho_2^2 e^{-2s\alpha}\xi^4) \cdot \nabla z_1 \Delta z_1 dxdt - s^2 \int_0^T \int_{\omega_4} \rho_2^2 e^{-2s\alpha}\xi^4 \nabla \Delta z_1 \cdot \nabla z_1 dxdt.$$

Using the estimate

$$|\nabla(\rho_2^2 e^{-2s\alpha}\xi^4)| \leq Cse^{-2s\alpha}\xi^5 \rho_2, \quad s \geq C,$$

and again Young's inequality for the first term, we obtain

$$s^2 \int_0^T \int_{\omega_3} e^{-2s\alpha}\xi^4 |\Delta z_1|^2 dxdt$$
$$\leq C\Big(\underbrace{s^4 \int_0^T \int_{\omega_4} e^{-2s\alpha}\xi^6 |\nabla z_1|^2 dxdt}_{I_1} - \underbrace{s^2 \int_0^T \int_{\omega_4} \rho_2^2 e^{-2s\alpha}\xi^4 \nabla \Delta z_1 \cdot \nabla z_1 dxdt}_{I_2} \Big) \tag{3.30}$$

for every $s \geq C$.

Now, to estimate I_1 we consider w_5 an open subset such that $w_4 \Subset w_5 \subset \omega'$ and $\rho_3 \in C_c^2(w_5)$ with $\rho_3 \equiv 1$ in ω_4 and $\rho_3 \geq 0$. Then

$$I_1 \leq s^4 \int_0^T \int_{\omega_5} \rho_3 e^{-2s\alpha}\xi^6 |\nabla z_1|^2 dxdt$$
$$\leq C\Big(s^6 \int_0^T \int_{\omega_5} e^{-2s\alpha}\xi^8 |z_1|^2 dxdt + s^4 \int_0^T \int_{\omega_5} \rho_3 e^{-2s\alpha}\xi^6 |\Delta z_1||z_1| dxdt \Big)$$
$$= C\Big(s^6 \int_0^T \int_{\omega_5} e^{-2s\alpha}\xi^8 |z_1|^2 dxdt + s^4 \int_0^T \int_{\omega_5} \rho_3 \theta^* |\Delta z_1| e^{-2s\alpha}(\theta^*)^{-1}\xi^6 |z_1| dxdt \Big),$$

for every $s \geq C$. We recall that $\theta^* := s^{1/2} e^{-s\alpha^*}(\xi^*)^{9/22}$.

Using Young's inequality for the second term we obtain for every $\varepsilon > 0$:

$$I_1 \leq \Big(s^6 \int_0^T \int_{\omega_5} e^{-2s\alpha}\xi^8 |z_1|^2 dxdt + \varepsilon \int_0^T \int_{\omega_5} |\theta^* \Delta z_1|^2 dxdt + C(\varepsilon) s^7 \int_0^T \int_{\omega_5} e^{-4s\alpha + 2s\alpha^*}\xi^{12} |z_1|^2 dxd \Big), \tag{3.31}$$

for every $s \geq C$. The second term in the right-hand side of the above inequality can be absorbed by the left-hand side of (3.27).

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Now we estimate I_2. An integration by parts gives

$$I_2 \leq C\Big(s^3 \int_0^T \int_{\omega_4} e^{-2s\alpha}\xi^5 |\nabla\Delta z_1||z_1|dxdt + s^2 \int_0^T \int_{\omega_4} e^{-2s\alpha}\xi^4 |\Delta^2 z_1||z_1|dxdt\Big).$$

Using again the Young's inequality, we obtain by an analogous argument the estimate:

$$I_2 \leq C\Big(\varepsilon\|\hat{\theta} z_1\|^2_{L^2(0,T;H^4(\omega_4))} + C(\varepsilon)s^5 \int_0^T \int_{\omega_4} e^{-4s\alpha+2s\alpha^*}\xi^{10}|z_1|^2 dxdt\Big), \tag{3.32}$$

for every $\varepsilon > 0$ and $s \geq C$. The first term in the right-hand side of (3.32) can be absorbed by the left-hand side of (3.27).

Finally, using the definition of the weight functions and (3.11), we readily obtain

$$\begin{aligned} & s^7 \int_0^T \int_{\omega_5} e^{-4s\alpha+2s\alpha^*}\xi^{12}|z_1|^2 dxdt \\ & \leq 2s^7 \int_0^T \int_{\omega_5} e^{-4s\hat{\alpha}+2s\alpha^*}(\hat{\xi})^{12}|\rho|^2|\varphi_1|^2 dxdt + 2s^7 \int_0^T \int_{\omega_5} e^{-4s\hat{\alpha}+2s\alpha^*}(\hat{\xi})^{12}|w_1|^2 dxdt \\ & \leq 2s^7 \int_0^T \int_{\omega_5} e^{-4s\hat{\alpha}+2s\alpha^*}(\hat{\xi})^{12}|\rho|^2|\varphi_1|^2 dxdt + C\|\rho g\|^2_{L^2(Q)^2}. \end{aligned}$$

From (3.27) and (3.28)–(3.32), we conclude the proof of Proposition 3.1.

4. Null controllability of the linear system

Here we are concerned with the null controllability of the following system:

$$\begin{cases} y_t - \nabla\cdot(Dy) + \nabla p = h + v\chi_\omega & \text{in} \quad Q, \\ \nabla\cdot y = 0 & \text{in} \quad Q, \\ y\cdot n = 0,\ (\sigma(y,p)\cdot n)_{tg} + (A(x,t)y)_{tg} = 0 & \text{on} \quad \Sigma, \\ y(\cdot,0) = y_0(\cdot) & \text{in} \quad \Omega, \end{cases} \tag{4.1}$$

where $y_0 \in W$, h is in an appropriate weighted space. We look for a control $v \in L^2(0,T;H^2(\omega)^N) \cap H^1(0,T;L^2(\omega)^N)$ such that $v_i \equiv 0$ for some $i \in \{1,\dots,N\}$.

To do this, let us first state a Carleman inequality with weight functions not vanishing in $t = 0$.

Let $\ell \in C^2([0,T])$ be a positive function in $[0,T)$ such that $\ell(t) > t(T-t)$ for all $t \in [0,T/4]$ and $\ell(t) = t(T-t)$ for all $t \in [T/2,T]$.

Now, we introduce the following weight functions:

$$\begin{aligned} &\beta(x,t) = \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x)}}{\ell^{11}(t)}, && \gamma(x,t) = \frac{e^{\lambda\eta(x)}}{\ell^{11}(t)}, \\ &\beta^*(t) = \max_{x\in\overline{\Omega}} \beta(x,t), && \gamma^*(t) = \min_{x\in\overline{\Omega}} \gamma(x,t), \\ &\widehat{\beta}(t) = \min_{x\in\overline{\Omega}} \beta(x,t), && \widehat{\gamma}(t) = \max_{x\in\overline{\Omega}} \gamma(x,t). \end{aligned} \tag{4.2}$$

Lemma 4.1. *Let $i \in \{1, \dots, N\}$ and let s and λ be like in Proposition 3.1. Then, there exists a constant $C > 0$ (depending on s and λ and increasing on $\|A\|_{P^1_\varepsilon \cap P^2}$) such that every solution φ of (3.2) satisfies:*

$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega)^N} + \iint\limits_Q e^{-6s\beta^*}(\gamma^*)^3|\varphi|^2 dxdt \\ \leq C\Big(\iint\limits_Q e^{-4s\beta^*}|g|^2 dxdt + \sum_{j=1, j\neq i}^N \int_0^T \int_\omega e^{-4s\hat{\beta}-2s\beta^*}(\hat{\gamma})^{12}|\chi_\omega \varphi_j|^2 dxdt\Big). \tag{4.3}$$

Proof. We start by an a priori estimate for the Stokes system (3.2). To do this, we introduce a function $\nu \in C^1([0,T])$ such that

$$\nu \equiv 1 \quad \text{in } [0, T/2], \qquad \nu \equiv 0 \quad \text{in } [3T/4, T].$$

We easily see that $(\nu\varphi, \nu\pi)$ satisfies

$$\begin{cases} -(\nu\varphi)_t - \nabla\cdot(D\nu\varphi) + \nabla(\nu\pi) = \nu g - \nu'\varphi & \text{in} \quad Q, \\ \nabla\cdot(\nu\varphi) = 0 & \text{in} \quad Q, \\ (\nu\varphi)\cdot n = 0,\ (\sigma(\nu\varphi,\nu\pi)\cdot n)_{tg} + (A^t(x,t)\nu\varphi)_{tg} = 0 & \text{on} \quad \Sigma, \\ (\nu\varphi)(T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{4.4}$$

Using (2.2) we have in particular

$$\|\varphi\|_{L^2(0,T/2;L^2(\Omega)^N)} + \|\varphi(\cdot,0)\|_{L^2(\Omega)^N} \\ \leq Ce^{CT\|A\|^2_{P^0_\varepsilon}}\Big(1 + \|A\|^2_{P^0_\varepsilon}\Big)\Big(\|g\|_{L^2(0,3T/4;L^2(\Omega)^N)} + \|\varphi\|_{L^2(T/2,3T/4;L^2(\Omega)^N)}\Big).$$

Taking into account that

$$e^{-4s\beta^*} \geq C > 0 \ \ \forall t \in [0, 3T/4] \quad \text{and} \quad e^{-6s\beta^*}(\gamma^*)^3 \geq C > 0,\ \ \forall t \in [T/2, 3T/4],$$

we have

$$\int_0^{T/2}\int_\Omega e^{-6s\beta^*}(\gamma^*)^3|\varphi|^2 dxdt + \|\varphi(\cdot,0)\|^2_{L^2(\Omega)^N} \\ \leq Ce^{CT\|A\|^2_{P^0_\varepsilon}}\Big(1 + \|A\|^2_{P^0_\varepsilon}\Big)\Big(\int_0^{3T/4}\int_\Omega e^{-4s\beta^*}|g|^2 dxdt + \int_{T/2}^{3T/4}\int_\Omega e^{-6s\beta^*}(\gamma^*)^3|\varphi|^2 dxdt\Big). \tag{4.5}$$

Note that, since $\alpha = \beta$ in $\Omega \times (T/2, T)$, we have:

$$\int_{T/2}^{T}\int_\Omega e^{-6s\beta^*}(\gamma^*)^3|\varphi|^2 dxdt = \int_{T/2}^{T}\int_\Omega e^{-6s\alpha^*}(\xi^*)^3|\varphi|^2 dxdt$$

and by virtue of Carleman inequality (3.3) (see Proposition 3.1), we obtain

$$\int_{T/2}^{T}\int_\Omega e^{-6s\beta^*}(\gamma^*)^3|\varphi|^2 dxdt \leq C\Big(\iint\limits_Q e^{-4s\alpha^*}|g|^2 dxdt + \sum_{j=1, j\neq i}^N \int_0^T\int_{\omega'} e^{-4s\hat{\alpha}-2s\alpha^*}(\hat{\xi})^{12}|\varphi_j|^2 dxdt\Big).$$

Since $\ell(t) = t(T-t)$ for any $t \in [T/2, T]$ and

$$e^{-4s\beta^*} \geq C \quad \text{and} \quad e^{-4s\hat{\beta}^* - 2s\beta^*}(\hat{\gamma})^{12} \geq C \quad \text{in } [0, T/2],$$

we readily get

$$\int_{T/2}^{T}\int_{\Omega} e^{-6s\beta^*}(\gamma^*)^3|\varphi|^2dxdt \leq C\Big(\iint_Q e^{-4s\beta^*}|g|^2dxdt + \sum_{j=1, j\neq i}^{N}\int_0^T\int_\omega e^{-4s\hat{\beta}-2s\beta^*}(\hat{\gamma})^{12}|\chi_\omega\varphi_j|^2dxdt\Big). \tag{4.6}$$

From (4.5) and (4.6) we obtain (4.3). □

Remark 4.1. Observe that on the left-hand side of (4.3) it is possible to put the terms

$$\|e^{-3s\beta^*}(\gamma^*)^{9/22}\varphi\|^2_{L^2(0,T;H^2(\Omega)^N\cap W)} + \iint_Q e^{-6s\beta^*}(\gamma^*)^{9/11}|\varphi_t|^2dxdt. \tag{4.7}$$

To this end, we consider $\tilde{\theta} := e^{-3s\beta^*}(\gamma^*)^{9/22}$ and $(\tilde{\theta}\varphi, \tilde{\theta}\pi)$ the solution of (4.4) with $\tilde{\theta}$ instead of ν. Next, taking into account that $|\partial_t\beta^*| \leq C(\gamma^*)^{12/11}$, $|\tilde{\theta}'| \leq Ce^{-3s\beta^*}(\gamma^*)^{3/2}$ and the regularity estimate (2.2), we obtain (4.7).

Now we are ready to prove the null controllability of system (4.1). The idea is to look for a solution in an appropriate weighted functional space. Let us set

$$Ly = y_t - \nabla \cdot Dy$$

and let us introduce the space, for $N = 2$ or $N = 3$ and $i \in \{1, \dots, N\}$,

$$\begin{aligned} E^i_N :=& \{(y, p, v) : e^{2s\beta^*}y, e^{2s\hat{\beta}+s\beta^*}(\hat{\gamma})^{-6}v,\ \tilde{\rho}\partial_t v \in L^2(Q)^N,\ \tilde{\rho}v \in L^2(0,T; H^2(\Omega)^N), \\ & v_i \equiv 0, supp\, v \subset \omega \times (0,T),\ e^{2s\beta^*}(\gamma^*)^{-12/11}y \in Y_1,\ e^{3s\beta^*}(\gamma^*)^{-3/2}(Ly + \nabla p - v\chi_\omega) \in L^2(Q)^N\}, \end{aligned}$$

where

$$\rho := e^{-4s\hat{\beta}-2s\beta^*}(\hat{\gamma})^{12} \quad \text{and} \quad \tilde{\rho} := \rho^{-1}\tilde{\theta}.$$

It is clear that E^i_N is a Banach space for the following norm:

$$\begin{aligned} \|(y,p,v)\|_{E^i_N} = \Big(&\|e^{2s\beta^*}y\|^2_{L^2(Q)^N} + \|e^{2s\hat{\beta}+s\beta^*}(\hat{\gamma})^{-6}v\|^2_{L^2(Q)^N} + \|\tilde{\rho}\partial_t v\|^2_{L^2(Q)} \\ &+ \|\tilde{\rho}v\|^2_{L^2(0,T;H^2(\Omega)^N)} + \|e^{2s\beta^*}(\gamma^*)^{-12/11}y\|^2_{Y_1} \\ &+ \|e^{3s\beta^*}(\gamma^*)^{-3/2}(Ly + \nabla p - v\chi_\omega)\|^2_{L^2(Q)^N}\Big)^{1/2}. \end{aligned}$$

Remark 4.2. Observe in particular that $(y, p, v) \in E^i_N$ implies $y(\cdot, T) = 0$ in Ω.

Proposition 4.1. *Assume the hypothesis of Lemma 4.1 and*

$$y_0 \in W \quad \textit{and} \quad e^{3s\beta^*}(\gamma^*)^{-3/2}h \in L^2(Q)^N. \tag{4.8}$$

Then, we can find a control v such that the associated solution (y,p) to (4.1) satisfies $(y,p,v) \in E_N^i$. In particular, $v_i \equiv 0$ and $y(\cdot,T) = 0$ in Ω. Furthermore, there exists $C > 0$ increasing with respect to $\|A\|_{P_\varepsilon^1 \cap P^2}$ such that

$$\|v\|_{L^2(0,T;H^2(\omega)^N)} + \|v\|_{H^1(0,T;L^2(\omega)^N)} \leq C\Big(\|y_0\|_{H^3(\Omega)^N \cap W} + \|h\|_{L^2(Q)^N}\Big). \tag{4.9}$$

Proof. Following the arguments in [11], we introduce the space P_0 of functions $(\varphi,\pi) \in C^2(\overline{Q})^{N+1}$ such that

(i) $\nabla \cdot \varphi = 0$ in Q.
(ii) $(\sigma(\varphi,\pi)\cdot n)_{tg} + (A^t(x,t)\varphi)_{tg} = 0$ on Σ.
(iii) $\varphi \cdot n = 0$ on Σ.

Also we define the bilinear form

$$a((\hat\varphi,\hat\pi),(w,q)) := \iint\limits_Q e^{-4s\beta^*}(L^*\hat\varphi + \nabla\hat\pi)(L^*w + \nabla q)dxdt + \sum_{j=1,j\neq i}^N \int_0^T \int_\omega e^{-4s\hat\beta - 2s\beta^*}(\hat\gamma)^{12}\chi_\omega\hat\varphi_j\chi_\omega w_j dxdt,$$

for every $(w,q) \in P_0$, and a linear form

$$\langle G,(w,q)\rangle := \iint\limits_Q h\cdot w\, dxdt + \int_\Omega y_0(\cdot)\cdot w(\cdot,0)\,dx, \tag{4.10}$$

where L^* is the adjoint operator of L, i.e.,

$$L^*w = -w_t - \nabla\cdot Dw.$$

Observe that Carleman inequality (4.3) holds for all $(w,q) \in P_0$. Consequently,

$$\iint\limits_Q e^{-6s\beta^*}(\gamma^*)^3|w|^2 dxdt \leq Ca((w,q),(w,q)), \quad \forall (w,q) \in P_0.$$

Therefore, $a(\cdot,\cdot) : P_0 \times P_0 \to \mathbb{R}$ is a symmetric, definite positive bilinear form on P_0. We denote by P the completion of P_0 for the norm induced by $a(\cdot,\cdot)$. Then, $a(\cdot,\cdot)$ is well-defined, continuous and again definite positive on P. Furthermore, in view of the Carleman inequality (4.3) and the assumption (4.8), the linear form $(w,q) \longmapsto \langle G,(w,q)\rangle$ is well-defined and continuous on P. Hence, from Lax–Milgram's Lemma, there exists one and only one $(\hat\varphi,\hat\pi) \in P$ satisfying:

$$a((\hat\varphi,\hat\pi),(w,q)) = \langle G,(w,q)\rangle, \quad \forall (w,q) \in P. \tag{4.11}$$

Let us set

$$\begin{cases} \hat y = e^{-4s\beta^*}(L^*\hat\varphi + \nabla\hat\pi) \quad \text{in } Q, \\ \hat v_j = -e^{-4s\hat\beta - 2s\beta^*}(\hat\gamma)^{12}\hat\varphi_j\chi_\omega, \ j \neq i, \ \hat v_i \equiv 0 \text{ in } \omega\times(0,T). \end{cases} \tag{4.12}$$

Let us remark that $(\hat{y}, \hat{v})$ verifies

$$a((\hat{\varphi},\hat{\pi}),(\hat{\varphi},\hat{\pi})) = \iint_Q e^{-4s\beta^*}(L^*\hat{\varphi}+\nabla\hat{\pi})^2 dxdt + \sum_{j=1,j\neq i}^N \int_0^T\int_\omega e^{-4s\hat{\beta}-2s\beta^*}(\hat{\gamma})^{12}|\chi_\omega\hat{\varphi}_j|^2 dxdt$$

$$= \iint_Q e^{4s\beta^*}|\hat{y}|^2 dxdt + \sum_{j=1,j\neq i}^N \int_0^T\int_\omega e^{4s\hat{\beta}+2s\beta^*}(\hat{\gamma})^{-12}|\hat{v}_j|^2 dxdt < +\infty.$$

Let us prove that $\hat{y}$ is, together with some pressure $\hat{p}$, the weak solution of the Stokes system in (4.1) for $v = \hat{v}$. In fact, we introduce the (weak) solution $(\tilde{y}, \tilde{p})$ to the Stokes system:

$$\begin{cases} L\tilde{y}+\nabla\tilde{p} = h+\hat{v}\chi_\omega & \text{in} \quad Q, \\ \nabla\cdot\tilde{y} = 0 & \text{in} \quad Q, \\ \tilde{y}\cdot n = 0,\ (\sigma(\tilde{y},\tilde{p})\cdot n)_{tg} + (A(x,t)\tilde{y})_{tg} = 0 & \text{on} \quad \Sigma, \\ \tilde{y}(\cdot,0) = y_0(\cdot) & \text{in} \quad \Omega. \end{cases} \tag{4.13}$$

Clearly, $\tilde{y}$ is the unique solution of (4.13) defined by transposition. This means that $\tilde{y}$ is the unique function in $L^2(Q)^N$ satisfying

$$\iint_Q \tilde{y}\cdot g\,dxdt = \int_\Omega y_0(\cdot)\cdot w(\cdot,0)dx + \iint_Q h\cdot w\,dxdt + \iint_Q \hat{v}\cdot w\chi_\omega\,dxdt, \quad \forall g\in L^2(Q)^N, \tag{4.14}$$

where w is, together with a pressure q, the solution to

$$\begin{cases} L^*w+\nabla q = g & \text{in} \quad Q, \\ \nabla\cdot w = 0 & \text{in} \quad Q, \\ w\cdot n = 0, (\sigma(w,q)\cdot n)_{tg} + (A^t(x,t)w)_{tg} = 0 & \text{on} \quad \Sigma, \\ w(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases}$$

From (4.11) and (4.12), we see that $\hat{y}$ also satisfies (4.14). Consequently, $\hat{y} = \tilde{y}$ and $\hat{y}$ is, together with $\hat{p} = \tilde{p}$, the weak solution to the Stokes system (4.13).

Finally, we must see that $(\hat{y}, \hat{p}, \hat{v}) \in E_N^i$. We already know that

$$e^{2s\beta^*}\hat{y},\ e^{2s\hat{\beta}+s\beta^*}(\hat{\gamma})^{-6}\hat{v} \in L^2(Q)^N$$

and (see (4.8))

$$e^{3s\beta^*}(\gamma^*)^{-3/2}(L\hat{y}+\nabla\hat{p}-\hat{v}\chi_\omega) \in L^2(Q)^N.$$

Thus, it only remains to check that

$$e^{2s\beta^*}(\gamma^*)^{-12/11}\hat{y}\in Y_1 \quad \text{and} \quad \tilde{\rho}\hat{v}\in L^2(0,T;H^2(\omega)^N)\cap H^1(0,T;L^2(\omega)^N).$$

i) We define the functions

$$y^* := e^{2s\beta^*}(\gamma^*)^{-12/11}\hat{y}, \quad p^* := e^{2s\beta^*}(\gamma^*)^{-12/11}\hat{p}$$

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and

$$h^* := e^{2s\beta^*}(\gamma^*)^{-12/11}(h + \hat{v}\chi_\omega).$$

Then (y^*, p^*) satisfies:

$$\begin{cases} Ly^* + \nabla p^* = h^* + (e^{2s\beta^*}(\gamma^*)^{-12/11})'\hat{y} & \text{in} \quad Q, \\ \nabla \cdot y^* = 0 & \text{in} \quad Q, \\ y^* \cdot n = 0, \ (\sigma(y^*, p^*) \cdot n)_{tg} + (A(x,t)y^*)_{tg} = 0 & \text{on} \quad \Sigma, \\ y^*(\cdot, 0) = e^{2s\beta^*(0)}(\gamma^*(0))^{-12/11} y_0(\cdot) & \text{in} \quad \Omega. \end{cases}$$

Since $h^* + (e^{2s\beta^*}(\gamma^*)^{-12/11})'\hat{y} \in L^2(Q)^N$ and $y_0 \in W$, we have $y^* \in Y_1$ (see Lemma 2.1 in Section 2).

ii) Now, let us bound the $H^1(0,T; L^2(\omega)^N)$ and the $L^2(0,T; H^2(\omega)^N)$ norms of the control. Using (4.12), we obtain

$$\sum_{j=1, j\neq i}^{N} \int_0^T \tilde{\rho}^2 (\|\partial_t \hat{v}_j\|^2_{L^2(\omega)} + \|\hat{v}_j\|^2_{H^2(\omega)}) dx dt$$
$$\leq C \sum_{j=1, j\neq i}^{N} \Big(\iint_Q e^{-6s\beta^*}(\gamma^*)^3 |\hat{\varphi}_j|^2 dx dt + \iint_Q \tilde{\theta}^2 |\partial_t \hat{\varphi}_j|^2 dx dt + \|\tilde{\theta}\hat{\varphi}_j\|^2_{L^2(0,T;H^2(\Omega))} \Big).$$

Taking into account that (4.3) and Remark 4.1 hold for all $(\hat{\varphi}, \hat{\pi}) \in P_0$, we readily obtain

$$\sum_{j=1, j\neq i}^{N} \int_0^T \tilde{\rho}^2 (\|\partial_t \hat{v}_j\|^2_{L^2(\omega)} + \|\hat{v}_j\|^2_{H^2(\omega)}) dx dt \leq C a((\hat{\varphi}, \hat{\pi}), (\hat{\varphi}, \hat{\pi})). \tag{4.15}$$

Finally, from the continuity of G (see (4.10)) and (4.11), we deduce (4.9). This ends the proof of Proposition 4.1. □

5. Proof of the main result

In this section we give the proof of Theorem 1.1 using classical arguments. The first step is to apply Kakutani's fixed point theorem on the boundary. Finally, we will deal with the nonlinear term in the Navier–Stokes equations through an inverse mapping theorem to conclude the proof of Theorem 1.1.

5.1. Nonlinearity on the boundary conditions

In this section we present the local null controllability for the following system:

$$\begin{cases} y_t - \nabla \cdot (Dy) + \nabla p = h + v\chi_\omega & \text{in} \quad Q, \\ \nabla \cdot y = 0 & \text{in} \quad Q, \\ y \cdot n = 0, \ (\sigma(y,p) \cdot n)_{tg} + (f(y))_{tg} = 0 & \text{on} \quad \Sigma, \\ y(\cdot, 0) = y_0(\cdot) & \text{in} \quad \Omega. \end{cases} \tag{5.1}$$

Theorem 5.1. *Let us assume that $f \in C^4(\mathbb{R}^N; \mathbb{R}^N)$ and $f(0) = 0$. Then, for every $T > 0$, $\omega \subset \Omega$ and $i \in \{1, \dots, N\}$, there exists $\delta > 0$ such that, for every $y_0 \in H^3(\Omega)^N \cap W$, $h \in Y_1$ satisfying $e^{3s\beta^*}(\gamma^*)^{-3/2} h \in L^2(Q)^N$,*

$$\|h\|_{Y_1} + \|y_0\|_{H^3(\Omega)^N \cap W} \leq \delta \tag{5.2}$$

and the compatibility condition (1.2)*, we can find a control*

$$v \in L^2(0,T;H^2(\omega)^N) \cap H^1(0,T;L^2(\omega)^N)$$

and an associated solution (y,p) *of* (5.1) *satisfying* $y \in Y_2$ *and such that* $(y,p,v) \in E_N^i$*.*

Proof. For every $z \in Z_\varepsilon$ (recall that Z_ε was defined in (1.4)) we consider the following system:

$$\begin{cases} y_t - \nabla \cdot (Dy) + \nabla p = h + v\chi_\omega & \text{in} \quad Q, \\ \nabla \cdot y = 0 & \text{in} \quad Q, \\ y \cdot n = 0,\ (\sigma(y,p)\cdot n)_{tg} + (g(z)y)_{tg} = 0 & \text{on} \quad \Sigma, \\ y(\cdot,0) = y_0(\cdot) & \text{in} \quad \Omega, \end{cases} \tag{5.3}$$

where

$$g(z) := \frac{1}{N}\int_0^1 \nabla f(\tau z)d\tau.$$

On the other hand, observe that since $f \in C^4(\mathbb{R}^N;\mathbb{R}^N)$, each row and each column of $g(z)$ belongs to Z_ε. Then, for every $z \in Z_\varepsilon$ we can use Proposition 4.1 with $A = g(z)$ and deduce the existence of a control v_z belonging to $L^2(0,T;H^2(\omega)^N) \cap H^1(0,T;L^2(\omega)^N)$ such that the solution (y_z,p_z) of (5.3) satisfies $(y_z,p_z,v_z) \in E_N^i$.

Moreover, from (4.9) we have

$$\|v_z\|_{L^2(0,T;H^2(\omega)^N)} + \|v_z\|_{H^1(0,T;L^2(\omega)^N)} \leq C_1(\Omega,\omega,T,\|g(z)\|_{P_\varepsilon^1\cap P^2})\Big(\|y_0\|_{H^3(\Omega)^N\cap W} + \|h\|_{L^2(Q)^N}\Big), \tag{5.4}$$

where C_1 is increasing with respect to $\|g(z)\|_{P_\varepsilon^1\cap P^2}$.

Next, taking into account that $v_z, h \in Y_1$ and the compatibility condition (2.3) with u_0 replaced by y_0, $A(\cdot,0)$ replaced by $g(y_0(\cdot))$ and $f_2(\cdot,0)$ replaced by 0 (see (1.2)), we can apply Theorem 2.1 to system (5.3). Combining this with (5.4), we can obtain that $y_z \in Y_2$ and

$$\|y_z\|_{Y_2} \leq C_2(\Omega,\omega,T,\|g(z)\|_{P_\varepsilon^1\cap P^2})\Big(\|y_0\|_{H^3(\Omega)^N\cap W} + \|h\|_{Y_1}\Big), \tag{5.5}$$

with C_2 increasing with respect to $\|g(z)\|_{P_\varepsilon^1\cap P^2}$ (see (2.6)).

Let $\mathcal{C}(z)$ be the set constituted by the controls $v_z \in L^2(0,T;H^2(\omega)^N) \cap H^1(0,T;L^2(\omega)^N)$ that satisfy (5.4) and drive the solution y_z of system (5.3) to zero at time T. Then, let us introduce

$$\Lambda(z) := \{y_z \text{ solution of (5.3) } : v_z \in \mathcal{C}(z)\}.$$

Observe that, thanks to (5.5), $\Lambda(z)$ is included in Y_2. Moreover, for any $z \in Y_2$ such that $\|z\|_{Y_2} \leq 1$, we have $\|g(z)\|_{P_\varepsilon^1\cap P^2} \leq M$, where $M > 0$ is a constant only depending on ε, T and Ω. Consequently,

$$\|y_z\|_{Y_2} \leq C_2(\Omega,\omega,T,M)\Big(\|y_0\|_{H^3(\Omega)^N\cap W} + \|h\|_{Y_1}\Big)$$

(see (5.5)). Choosing now $\delta := \dfrac{1}{C_2(\Omega,\omega,T,M)}$ in (5.2), we find $\|y_z\|_{Y_2} \leq 1$.

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Now, we want to establish that the set-valued map $\Lambda : K \to 2^K$ possesses a fixed-point, where

$$K := \overline{B}_{Y_2}(0;1) = \{y \in Y_2 : \|y\|_{Y_2} \le 1\}.$$

For this end, we will apply Kakutani's fixed-point theorem (see for instance [1], Theorem 3.2.3, page 87):

i) $\Lambda(z)$ is a nonempty closed convex set of $L^2(Q)^N$, for every $z \in K$.
ii) K is a nonempty convex compact set of $L^2(Q)^N$.
iii) Λ is upper-hemicontinuous in $L^2(Q)^N$, i.e., for any $\lambda \in L^2(Q)^N$, the mapping

$$z \to \sup_{y\in\Lambda(z)} \langle \lambda, y\rangle_{L^2(Q)^N}$$

is upper semicontinuous.

i) For every $z \in K$, let $(y_z^k) \subset \mathcal{C}(z)$ such that $y_z^k \to y_z$ in $L^2(Q)^N$. From (5.4), we find (at least for a subsequence) that $v_z^{k'} \to v_z$ in $L^2(Q)^N$. Let us denote w_z the solution of (5.3) associated to $v := v_z$. Then, $y_z^{k'} - w_z$ satisfies (5.3) with $h := 0$, $v := v_z^{k'} - v_z$ and $y_0 := 0$. Thanks to (2.2), we have $y_z^{k'} \to w_z$ in $L^2(Q)^N$ in particular and so $y_z = w_z$. This shows that $\Lambda(z)$ is closed. The convexity of $\Lambda(z)$ is trivial.
ii) Since Y_2 is compactly embedded into $L^2(Q)^N$, the second item holds true.
iii) Finally, let us prove the upper-hemicontinuity of Λ. Assume $z_k \to z$ in $L^2(Q)^N$. In consequence from the compactness of $\Lambda(z_k)$, we have

$$\sup_{y\in\Lambda(z_k)} \langle \lambda, y\rangle_{L^2(Q)^N} = \langle \lambda, y_k\rangle_{L^2(Q)^N},$$

for some $y_k \in \Lambda(z_k)$. Then, we choose $(z_{k'}) \subset (z_k)$ such that

$$\lim_{k'\to\infty} \sup_{y\in\Lambda(z_{k'})} \langle \lambda, y\rangle_{L^2(Q)^N} = \lim_{k'\to\infty} \langle \lambda, y_{k'}\rangle_{L^2(Q)^N}$$

and denote $v_{k'}$ the controls in $\mathcal{C}(z_{k'})$ which are associated to $y_{k'} \in \Lambda(z_{k'})$. From (5.4), there exists $v^* \in L^2(0,T;H^2(\omega)^N) \cap H^1(0,T;L^2(\omega)^N)$ such that $v_{k'} \rightharpoonup v^*$ in $L^2(0,T;H^2(\omega)^N) \cap H^1(0,T;L^2(\omega)^N)$ and $v^* \in \mathcal{C}(z)$. In particular, $v_{k'} \to v^*$ in $L^2(Q)^N$ (for a subsequence). Now, let (y^*, p^*) be the solution to (5.3) associated to v^*. We set $\tilde{y}_{k'} := y_{k'} - y^*$, $\tilde{p}_{k'} := p_{k'} - p^*$ and $\tilde{v}_{k'} := v_{k'} - v^*$. Then,

$$\begin{cases} (\tilde{y}_{k'})_t - \nabla\cdot(D\tilde{y}_{k'}) + \nabla\tilde{p}_{k'} = \tilde{v}_{k'}\chi_\omega & \text{in} \quad Q, \\ \nabla\cdot\tilde{y}_{k'} = 0 & \text{in} \quad Q, \\ \tilde{y}_{k'}\cdot n = 0,\ (\sigma(\tilde{y}_{k'},\tilde{p}_{k'})\cdot n)_{tg} + (g(z)\tilde{y}_{k'})_{tg} = ([g(z)-g(z_{k'})]y_{k'})_{tg} & \text{on} \quad \Sigma, \\ \tilde{y}_{k'}(\cdot,0) = 0 & \text{in} \quad \Omega. \end{cases}$$

Taking into account that $g(z_{k'}) \to g(z)$ in Z_ε, one can prove that in particular

$$\|[g(z)-g(z_{k'})]y_{k'}\|_{L^2(0,T;H^{1/2}(\partial\Omega)^N)\cap H^{1/4+\varepsilon}(0,T;H^{-\varepsilon}(\partial\Omega)^N)} \xrightarrow{k'\to\infty} 0.$$

Then, from Lemma 2.1 we can deduce that $y_{k'} \to y^*$ in Y_1. Additionally, $y^* \in \Lambda(z)$ and therefore,

$$\lim_{k'\to\infty} \sup_{y\in\Lambda(z_{k'})} \langle \lambda, y\rangle_{L^2(Q)^N} = \lim_{k'\to\infty} \langle \lambda, y_{k'}\rangle_{L^2(Q)^N} = \langle \lambda, y^*\rangle_{L^2(Q)^N} \le \sup_{y\in\Lambda(z)} \langle \lambda, y\rangle.$$

This concludes the proof of Theorem 5.1. □

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5.2. Nonlinearity in the main equation

Theorem 5.2. *Suppose that $\mathcal{B}_1, \mathcal{B}_2$ are Banach spaces and*

$$\mathcal{A} : \mathcal{B}_1 \to \mathcal{B}_2$$

is a continuously differentiable map. We assume that for $b_1^0 \in \mathcal{B}_1, b_2^0 \in \mathcal{B}_2$ the equality

$$\mathcal{A}(b_1^0) = b_2^0 \tag{5.6}$$

holds and $\mathcal{A}'(b_1^0) : \mathcal{B}_1 \to \mathcal{B}_2$ is an epimorphism. Then there exists $\delta > 0$ such that for any $b_2 \in \mathcal{B}_2$ which satisfies the condition

$$\|b_2^0 - b_2\|_{\mathcal{B}_2} < \delta$$

there exists a solution $b_1 \in \mathcal{B}_1$ of the equation

$$\mathcal{A}(b_1) = b_2.$$

We apply this theorem for some given $i \in \{1, \dots, N\}$ and the spaces

$$\mathcal{B}_1 := \{(y, p, v) \in E_N^i : y \in Y_2\}$$

and

$$\mathcal{B}_2 := \{(h, y_0) \in [L^2(e^{3s\beta^*}(\gamma^*)^{-3/2}(0,T); L^2(\Omega)^N) \cap Y_1] \times [H^3(\Omega)^N \cap W] : h, y_0 \text{ satisfies } (5.2)\}.$$

We define the operator $\mathcal{A}$ by the formula

$$\mathcal{A}(y, p, v) = (Ly + (y \cdot \nabla)y + \nabla p - v\chi_\omega, y(\cdot, 0)).$$

Let us see that $\mathcal{A}$ is of class $C^1(\mathcal{B}_1, \mathcal{B}_2)$. Indeed, notice that all the terms in $\mathcal{A}$ are linear, except for $(y \cdot \nabla)y$. We prove now that the bilinear operator

$$((y^1, p^1, v^1), (y^2, p^2, v^2)) \longmapsto (y^1 \cdot \nabla)y^2$$

is continuous from $\mathcal{B}_1 \times \mathcal{B}_1$ to $L^2(e^{3s\beta^*}(\gamma^*)^{-3/2}(0,T); L^2(\Omega)^N) \cap Y_1$.

In fact, notice that (see the definition of the space E_N^i):

$$e^{2s\beta^*}(\gamma^*)^{-12/11} y \in L^2(0,T; L^\infty(\Omega)^N)$$

and

$$\nabla(e^{2s\beta^*}(\gamma^*)^{-12/11} y) \in L^\infty(0,T; L^2(\Omega)^{N\times N}).$$

Consequently, we obtain

$$\begin{aligned}
&\|e^{3s\beta^*}(\gamma^*)^{-3/2}(y^1 \cdot \nabla)y^2\|_{L^2(Q)^N} \\
&\le C\|(e^{2s\beta^*}(\gamma^*)^{-12/11}y^1 \cdot \nabla)e^{2s\beta^*}(\gamma^*)^{-12/11}y^2\|_{L^2(Q)^N} \\
&\le C\|e^{2s\beta^*}(\gamma^*)^{-12/11}y^1\|_{L^2(0,T;L^\infty(\Omega)^N)}\|e^{2s\beta^*}(\gamma^*)^{-12/11}y^2\|_{L^\infty(0,T;W)}.
\end{aligned}$$

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On the other hand,

$$\|(y^1 \cdot \nabla)y^2\|_{Y_1} \leq C\|y^1\|_{Y_2}\|y^2\|_{Y_2}.$$

Notice that $\mathcal{A}'(0,0,0) : \mathcal{B}_1 \to \mathcal{B}_2$ is given by

$$\mathcal{A}'(0,0,0)(y,p,v) = (Ly + \nabla p - v\chi_\omega, y(\cdot,0)), \quad \text{for all } (y,p,v) \in \mathcal{B}_1.$$

In virtue of Theorem 5.1, this functional satisfies $Im(\mathcal{A}'(0,0,0)) = \mathcal{B}_2$.

Let $b_1^0 = (0,0,0)$ and $b_2^0 = (0,0)$. Then equation (5.6) obviously holds. So all necessary conditions to apply Theorem 5.2 are fulfilled. Therefore there exists a positive number δ such that, if $\|y(\cdot,0)\|_{H^3(\Omega)^N \cap W} \leq \delta$, we can find a control v satisfying $v_i \equiv 0$, for some given $i \in \{1,\dots,N\}$ and an associated solution (y,p) to (1.1) satisfying $y(\cdot,T) = 0$ in Ω. This finishes the proof of Theorem 1.1.

6. Comments and open problems

One of the main novelties is Proposition 3.1, which involves new estimates for the pressure term from known Carleman inequalities for parabolic equations and a new regularity result for the Stokes system with linear Navier-slip conditions (see Theorem 2.1). Otherwise, when $N = 3$, the major difficulty by thinking on local null controllability of (1.1) with one single control is to estimate the global integrals of two components of velocity through a third component, which is not clear at all and therefore is an open problem.

On the other side, the local null controllability for the Boussinesq system with Dirichlet boundary conditions and $N-1$ scalar controls have been established by Carreño [3], so that, it is reasonable to expert results of the same kind whether instead of Dirichlet conditions one considers nonlinear Navier-slip conditions. Additionally, it would be interesting to study the extension of arguments exposed in this paper to other fluid models such as appears in micropolar fluids [10,13] and in a model of turbulence [14].

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References

[1] J-P. Aubin, H. Frankowska, Set-Valued Analysis, Birkhäuser, Boston, 1990.
[2] F. Boyer, P. Fabrie, Mathematical Tools for the Navier–Stokes Equations and Related Models Study of the Incompressible, Springer-Verlag, New York, 2013.
[3] N. Carreño, Local controllability of the N-dimensional Boussinesq system with $N-1$ scalar controls in an arbitrary control domain, preprint arXiv:1201.1871, 2012.
[4] N. Carreño, S. Guerrero, Local null controllability of the N-dimensional Navier–Stokes system with $N-1$ scalar controls in an arbitrary control domain, J. Math. Fluid Mech. 15 (1) (2013) 139–153.
[5] T. Cebeci, A.M. Smith, Analysis of Turbulent Boundary Layers, Applied Mathematics and Mechanics, vol. 15, Academic Press [Harcourt Brace Jovanovich, Publishers], New York–London, 1974.
[6] J.-M. Coron, S. Guerrero, Null controllability of the N-dimensional Stokes system with $N-1$ scalar controls, J. Differ. Equ. 246 (2009) 2908–2921.
[7] J.-M. Coron, On the controllability of the 2-D incompressible Navier–Stokes equations with the Navier slip boundary conditions, ESAIM Control Optim. Calc. Var. 1 (1995/96) 35–75.
[8] J.-M. Coron, P. Lissy, Local null controllability of the three-dimensional Navier–Stokes system with a distributed control having two vanishing components, Invent. Math. 198 (3) (2014) 833–880.
[9] J.-M. Coron, F. Marbach, F. Sueur, Small time global exact controllability of the Navier–Stokes equation with Navier slip-with-friction boundary conditions, preprint arXiv:1612.08087, 2016.
[10] P. Cornilleau, S. Guerrero, On the local exact controllability of micropolar fluids with few controls, ESAIM Control Optim. Calc. Var. 23 (2) (2017) 637–662.
[11] E. Fernández-Cara, S. Guerrero, O.Yu. Imanuvilov, J.-P. Puel, Local exact controllability of the Navier–Stokes system, J. Math. Pures Appl. 83 (2004) 1501–1542.

[12] E. Fernández-Cara, M. Gonzalez-Burgos, S. Guerrero, J.-P. Puel, Null controllability of the heat equation with boundary Fourier conditions: the linear case, ESAIM Control Optim. Calc. Var. 12 (3) (2006) 442–465.

[13] E. Fernández-Cara, S. Guerrero, Local exact controllability of micropolar fluids, J. Math. Fluid Mech. 9 (3) (2007) 419–453.

[14] E. Fernández-Cara, J. Límaco, S.B. de Menezes, Theoretical and numerical local null controllability of a Ladyzhenskaya–Smagorinsky model of turbulence, J. Math. Fluid Mech. 17 (4) (2015) 669–698.

[15] A. Fursikov, O.Yu. Imanuvilov, Controllability of Evolution Equations, Lecture Notes, vol. 34, Seoul National University, Korea, 1996.

[16] S. Guerrero, Local exact controllability to the trajectories of the Navier–Stokes system with nonlinear Navier-slip boundary conditions, ESAIM Control Optim. Calc. Var. 12 (3) (2006) 484–544.

[17] O.Yu. Imanuvilov, Local exact controllability for the 2-D Navier–Stokes equations with the Navier slip boundary conditions, in: Turbulence Modelling and Vortex Dynamics, Istanbul, 1996, in: Lect. Notes Phys., vol. 491, Springer, Berlin, 1997, pp. 148–168.

[18] O.Yu. Imanuvilov, J.-P. Puel, Global Carleman estimates for weak elliptic nonhomogeneous Dirichlet problem, Int. Math. Res. Not. 16 (2003) 883–913.

[19] E. Lauga, M.P. Brenner, H.A. Stone, Microfluidics: the no-slip boundary condition, in: Handbook of Experimental Fluid Dynamics, Springer, 2006.

[20] J.-L. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, Volume I, Springer-Verlag, Berlin, Heidelberg, New York, 1972.

[21] C.-L. Navier, Mémoire sur les lois du mouvement des fluides, Mém. Acad. R. Sci. 6 (1823) 389–440.

[22] R. Temam, Navier Stokes Equations, Theory and Numerical Analysis, Stud. Math. Appl., vol. 2, North-Holland, Amsterdam, 1977.

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