INVERSE SOURCE PROBLEMS FOR COUPLED PARABOLIC SYSTEMS USING MEASUREMENTS OF ONE SCALAR STATE–PART I: THEORETICAL ANALYSIS

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Abstract. This paper is devoted to the study of inverse source problems for coupled systems of heat equations with constant or spatial–dependent coupling terms and whose internal measurements involve a reduced number of observed states. The analysis is developed for three kind of systems: the first one consists of parabolic equations with zero order coupling terms (or the so–called non–self–adjoint matrix potential) and whose possibly space–dependent coefficients. The second one consists of parabolic equations with coupling in the diffusion matrix. For these kinds of systems we establish source reconstruction formulas using internal measurements of one scalar state. The last one obeys the case of coupled non-linear systems of heat equations, where a Lipschitz–type stability is proven for the spatial factor in the source term using observation data on an arbitrary fixed sub–domain related to only one scalar state. In all configurations the source is decomposed in separate variables, where the temporal part is known and scalar, whereas the spatial dependence is an unknown vector field. This work builds on previous methodologies for the recovery of source in scalar equations and Stokes fluids, thus expanding the field to include coupled systems of second order parabolic equations.
Key words: Inverse problems, non–self–adjoint matrix Sturm–Liouville operators, null controllability problem, Volterra equations and Carleman estimates.

Contents

1. Main problems

1.1. **Introduction.** Inverse problems of determining coefficients or sources in coupled systems of partial differential equations has drawn an increasing interest during the last decade, specially in the case of parabolic or hyperbolic systems, although also in more complex systems that naturally appear in many branches of science and engineering, including fluid mechanics, biology, medicine, among others. A

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challenging task in inverse problems for coupled systems is related to whether it is possible to determine all sources (or coefficients) by a reduced numbers of measurements, where usually the measurements are given by all state variables on local subsets (either from boundary local subsets or internal local subsets), in other words, the issue of measuring a less quantity of states than the number of sources (resp. coefficients). This subject is interesting from both theoretical and practical point of view.

From a theoretical point of view, this question leads to different guidelines depending on types of coupling. For instance, linear coupling in low–order terms might show matrix potentials with non–self adjoint operators, where tools such as perturbation theory, semigroups theory and spectral analysis are frequently used in order to analyze those systems, or, even linear coupling in the main operator, which could easily lead to degenerate operators. Obviously, the analysis of systems with nonlinear coupling terms turns out to be more complex than the previous one, what in its turn it requires knowledge in approximation theory as well as fixed point arguments. To continue, let us mention that, as explained in [29], the reconstruction of a general external source either from internal or boundary measurements is not determined uniquely, thus, by adding external forces into coupled systems, the inverse source problem becomes solvable if some a priori knowledge is assumed, i.e., if the unknown source is a characteristic function [29], a point source [23], one part in the separation of variables [22].

Concerning practical applications, there exist models in the real life involving partial data of physical quantities, for example, pressure estimation from velocity phase–contrast MRI [11], wireless communication where only some components of the electric fields are measured [18], elastography [20], molecular multi–photon transitions in laser fields [8], heat transfer [2], hybrid inverse problems [7].

In the present study, the source is decomposed in separate variables, where the temporal part is known and scalar, meanwhile, the spatial dependence is an unknown vector field. In fact, these external forces are associated to coupled systems of heat equations. Let us emphasize that our precise goal in the present work will be to recover the spatial distribution of external forces in coupled parabolic systems from a reduced number of measured states in internal subsets. More precisely, our propose is to provide explicit formulas which show that it is possible to recover sources in coupled systems of heat equations from a limited number of component of the state in small subdomains. Here, the word "recover" refers to two issues: the first one is that the measurements determine a source reconstruction formula of the coefficients associated to f. The another one is to describe algorithms to compute the source terms. Furthermore, it is worth pointing out that this work is inspired on a previous methodology for the recover of sources in scalar equations (heat equation [27], wave equation [40]) and Stokes fluids [26], thus allowing us to transfer the existing results on scalar parabolic equations to system of heat equations.

To be more precise, let us describe our inverse source problems from an abstract framework. Throughout the paper, Ω will be a nonempty bounded domain of $\mathbb{R}^d$ ($d \in \mathbb{N}$) with smooth boundary $\partial\Omega$, $\mathcal{O} \subset \Omega$ a nonempty open subset, which will denote the spatial subset of measurements. We denote by $n \in \mathbb{N}$ the number of equations and by $m \in \mathbb{N}$ the number of observed components, with $m < n$. Let A be an appropriate linear partial differential operator with domain $D(A) \subset L^2(\Omega, \mathbb{K})$, time–independent, and (possibly) space–dependent coefficients, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The source term is of the form $\sigma(t)F(x)$, where σ is a known scalar function whereas $F = (f_1, \ldots, f_n)^*$ is unknown, and both in suitable spaces to define later on. In the present paper, we will focus on the following problems:

Problem 1. *The diagonal case with the same operator A on each line, that is,*

$$\begin{cases} \partial_t Y + I_n AY + QY = \sigma(t)F(x) & in \quad \Omega \times (0,T), \\ Y(\cdot, 0) = 0, & in \quad \Omega, \end{cases} \tag{1.1}$$

where $Q \in \mathcal{M}_n(\mathbb{K})$ is a coupling matrix with possibly space–dependent coefficients and I_n is the identity matrix of size n. Here, we are interested in solving the following question: can we recover the source term

$F = (f_1, \dots, f_n)^*$ in system (1.1) from incomplete data, that is, from a reduced numbers of measurements of the solution in $\mathcal{O} \times (0, T)$?

Problem 2. *The case where the coupling is in the principal part:*

$$\begin{cases} \partial_t Y + DAY = \sigma(t)F(x) & in \quad \Omega \times (0,T), \\ Y(\cdot, 0) = 0 & in \quad \Omega, \end{cases} \tag{1.2}$$

with $D \in \mathcal{M}_n(\mathbb{K})$ a diffusion matrix with constant coefficients. In this case, D is assumed to be diagonalizable with positive eigenvalues. To this case, our question is: can we recover the source term $F = (f_1, \dots, f_n)^*$ in system (1.2) from incomplete data, that is, from a reduced numbers of measurements of the solution in $\mathcal{O} \times (0, T)$?

Problem 3. *The case of coupled nonlinear systems of parabolic equations:*

$$\begin{cases} \partial_t Y + I_n AY + QY + G(Y) = \sigma(t)F(x) & in \quad \Omega \times (0,T), \\ Y(\cdot, 0) = 0, & in \quad \Omega, \end{cases} \tag{1.3}$$

where $Q \in \mathcal{M}_n(\mathbb{K})$ and $G(Y) = (g_1(Y), \dots, g_n(Y))^* \in \mathcal{M}_n(\mathbb{K})$ is assumed to be Lipschitz continuous with respect to the vector variable Y.

Now, we are interested in the following question: can we determine the source terms $F = (f_1, \dots, f_n)^*$ in system (1.3) from observation data $y_n\Big|_{\mathcal{O}\times(0,T)}$?

1.2. **State of the art.**

a) In relation to Problem 1 and under the structure of system (1.1) with $A = -\Delta$, the Bukhgeim–Klibanov method [31], which is based on Carleman estimates, have been employed to obtain some results in the context on inverse coefficient problems. For instance: the articles [15, 10] proved Lipschitz–type stability inequalities for 2×2 reaction–diffusion systems with a single observation acting on a subdomain. In addition, [16] gives an extension of those works to the nonlinear case. Nevertheless, by considering hyperbolic systems in cascade formed by n equations, the recent work [12] addresses uniqueness and stability (Lipschitz stability) aspects from internal measurements of all components of the solution except the last one. Another recent work is [19] for two coupled Schrödinger equations (that is, $A = i\Delta$ in (1.1)), where the authors treated the logarithmic stability for determining two potentials from internal observations of one component of the solution. In contrast to the previous results, [19] combines Carleman estimates along with the Fourier–Bros–Iagolnitzer transform in order to prove the logarithmic stability. The paper [14] deals the identification problem of two discontinuous coefficients for a one–dimensional coupled parabolic system, observing only one component. In [14] the authors proved Carleman–type inequalities for theoretical issues, whereas the numerical results have been carried out using the finite difference method for the temporal and spatial discretization jointly with the interior–point method (i.e., optimization problem).

In summary, the above articles show either identificability or stability properties for coupled hyperbolic or parabolic systems using incomplete measurements of their components. However, respect to inverse source problems, to the best of our knowledge, we have: by considering a cascade system of n degenerate parabolic equations with zero–order coupling terms and a general external force $G = G(x,t)$ instead of $\sigma(t)F(x)$, an identification problem is established in [3]. More precisely, under certain assumptions on G and its temporal derivative, the article [3] proves a Lipschitz–type stability (by means of Carleman inequalities) for determining the source G by observations in an subdomain of only one component and also using data of the n components at the fixed positive time over the whole spatial domain. The paper closest to our research is [1], which considers two wave equations coupled in cascade. In fact, the identification and stability problems of space–dependent sources from boundary incomplete observations are solved in [1]. Nevertheless, [1] does not provide insights upon exact reconstruction formulas to those sources. Explicit formulas of source reconstruction and their algorithms for coupled parabolic systems have not been reported in the literature. Therefore, the first purpose of this paper is to fill that gap for systems (1.1) where A is the Laplace operator.

b) On Problem 2, as far as we know, there exist few works concerning inverse problems for coupled systems such as (1.2). The recent paper [13] deals the coefficients determination problem for an transmitted diseases model (coupled parabolic equations in relation with the SIR model) whose coupling is located in the principal term associated to the operator. The authors of [13] applied optimal control techniques with constrains in order to formulate and prove their main result. Another recent work is [39], where the authors have proved Carleman inequalities to derive Hölder stability for the inverse coefficient problem from internal observation data for a three–dimensional system of two coupled heat equations with similar structure to (1.2).

c) In the spirit to Problem 3, inverse problems corresponding to coupled nonlinear systems have been less studied and the results obtained are of different natures. The following ones are worth mentioning: [16] analyzes the identification problem of two coefficients with data of one component for a 2×2–order nonlinear parabolic system similar to (1.3) (by means of Carleman estimates). As mentioned, [13] deals the coefficients determination problem for an transmitted diseases model throughout optimal control techniques.

To finish, we also refer to the articles [37, 38, 24] and their references therein, which are closely linked to inverse problems for coupled parabolic systems.

As mentioned, our approach follows the guidelines proposed in [40, 27] and [26] for the wave, heat and Stokes equations, respectively. Therefore, preliminary results on Volterra equations, as well as Riesz bases and controllability properties for coupled parabolic systems are given in section 2. In the remainder of this paper, our goal is threefold:

- In section 3 we present source reconstruction formulas for parabolic systems whose coupling is a constant coefficients matrix located either in the principal part of the operator or as a potential term. In other words, we solve Problems 1, 2 for systems with constant coefficients and using internal measurements in $\mathcal{O} \times (0,T)$ from only one scalar component of the state, i.e., through $y_n|_{\mathcal{O}\times(0,T)}$. In relation to Problem 1, our study is an extension of the results proven in [27] for the scalar case.
- In section 4 we give a source reconstruction formula for one dimensional systems of two heat equations with non–constant zero–order coupling term (i.e., potential matrix with space–dependent coefficients). Roughly speaking, we solve Problem 1 in 1D for a 2×2–order parabolic system with non–self–adjoint operators and from observation data $y_2|_{\mathcal{O}\times(0,T)}$. The main difficulty is that the main operator is not self–adjoint, and therefore properties on the eigenfunctions associated with non–self–adjoint matrix Sturm–Liouville operators are needed in order to provide a source reconstruction result from only interior measurements and a reduced number of states.
- In section 5 we give a positive answer related to Problem 3. More precisely, for the nonlinear system (1.3) we prove a Lipschitz–type stability for the spatial factor in the source term using observation data on an arbitrary fixed sub–domain over a time interval for the nth state y_n, that is, measurements on $y_n|_{\mathcal{O}\times(0,T)}$. Carleman estimates and the Bukhgeim–Klibanov method constitute the main tools in order to achieve such a property.

2. Preliminaries

As mentioned, in order to provide source reconstruction formulas for coupled systems of heat equations, we shall mainly combine existing null controllability results for coupled parabolic systems with distributed control, spectral properties for linear operators, and also integro–differential equations. Thus, this section is devoted to present these topics. Henceforth, Problems 1–3 will be analyzed by considering the operator $A = -\Delta$ with domain $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$.

2.1. **Spectral analysis.** In this paragraph we describe the eigenvalues and eigenfunctions of the non–self adjoint operators $L, L^* : D(L) = D(L^*) = H^2(0,\pi)^2 \cap H_0^1(0,\pi)^2 \subset L^2(0,\pi)^2 \to L^2(0,\pi)^2$ related to the following Sturm–Liouville problem

$$\begin{cases} L\Phi := -\Delta\Phi + V(x)\Phi = \lambda\Phi & \text{in } (0,\pi), \\ \Phi(0,t) = \Phi(\pi,t) = 0, \end{cases} \tag{2.1}$$

where

$$V(x) = \begin{pmatrix} 0 & 0 \\ q(x) & 0 \end{pmatrix}, \quad \text{and} \quad q \in L^\infty(0,\pi). \tag{2.2}$$

Additionally, for every $k \in \mathbb{N}$, we consider the normalized eigenfunctions of the Laplace operator with Dirichlet boundary conditions over $(0,\pi)$, i.e.,

$$\varphi_k(x) = \sqrt{\frac{2}{\pi}}\sin(kx), \quad \text{and also the expression} \quad I_k(q) := \int_0^\pi q(x)\varphi_k(x)dx, \quad \forall\, k \in \mathbb{N}. \tag{2.3}$$

The next Lemma establishes biorthogonal Riesz bases associated to the operators L and L^*. All details can be found in [21] and [6].

Lemma 1. *Consider the families*

$$\mathcal{B} = \left\{ \Phi_{1,k} = \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix}, \Phi_{2,k} = \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix} : k \in \mathbb{N} \right\} \quad \textit{and} \quad \mathcal{B}^* = \left\{ \Phi^*_{1,k} = \begin{pmatrix} \psi_k \\ \varphi_k \end{pmatrix}, \Phi^*_{2,k} = \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix}, : k \in \mathbb{N} \right\},$$

where ψ_k is defined for all $x \in (0,\pi)$ by

$$\begin{cases} \psi_k(x) = \alpha_k\varphi_k(x) - \dfrac{1}{k}\displaystyle\int_0^x \sin(k(x-\zeta))(I_k(q)\varphi_k(\zeta) - q(\zeta)\varphi_k(\zeta))d\zeta, \\ \alpha_k = \dfrac{1}{k}\displaystyle\int_0^\pi\int_0^x \sin(k(x-\zeta))((I_k(q)\varphi_k(\zeta) - q(\zeta)\varphi_k(\zeta)))\varphi_k(x)d\zeta dx. \end{cases} \tag{2.4}$$

Then, one has

a) *The spectrum of L^* and L are given by $\rho(L^*) = \rho(L) = \{k^2 : k \in \mathbb{N}\}$.*
b) *For every $k \in \mathbb{N}$, the eigenvalue k^2 of L^* has algebraic multiplicity 1. Moreover, in this case,*

$$\begin{cases} \left(L^* - k^2 Id\right)\Phi^*_{1,k} = I_k(q)\Phi^*_{2,k}, \\ \left(L^* - k^2 Id\right)\Phi^*_{2,k} = 0. \end{cases} \tag{2.5}$$

c) *For every $k \in \mathbb{N}$, the eigenvalue k^2 of L has algebraic multiplicity 1. Moreover, in this case,*

$$\begin{cases} \left(L - k^2 Id\right)\Phi_{1,k} = 0, \\ \left(L - k^2 Id\right)\Phi_{2,k} = I_k(q)\Phi_{1,k}. \end{cases} \tag{2.6}$$

d) *The sequences $\mathcal{B}$ and $\mathcal{B}^*$ are biorthogonal Riesz basis of $L^2(0,\pi)^2$.*
e) *The sequence $\mathcal{B}^*$ is a Schauder basis of $H_0^1(0,\pi)^2$ and $\mathcal{B}$ is its biorthogonal basis in $H^{-1}(0,\pi)^2$.*

Remark 1. *Note that Lemma 1 shows a condition on the potential $V(x)$ for which the root functions of the operators L and L^* form Riesz bases. Roughly speaking, the spectral theory for either regular or singular Sturm–Liouville problems offers a wide variety of topics in order to analyze its eigenvalues and eigenfunctions, see for instance* [33, 35, 30]. *However, we only incorporated the one–dimensional case of a 2×2–order system with non–self adjoint matrix potential, which allows us to understand the main purpose of the spectral part in our strategy as well as to indicate the key points for future works. But this is a very broad and independent area, which involves a delicate and deep analysis even in 1D for $n \times n$–order systems. We do not touch on this are in our paper. Some impression of these issues can be gained from the papers* [36, 17, 34].

2.2. **Controllability.** Depending on the structure of the coupling matrix Q^* and the control matrix B, different results on null controllability for the adjoint system of (1.1) can de derived, see for instance [28, 9, 25]. To our purpose, let us assume that $Q^* \in L^\infty(\Omega)^{n^2}$ and $B \in \mathcal{M}_n(\mathbb{R})$ have the structure:

$$Q^* = \begin{pmatrix} q_{11} & 0 & 0 & \cdots & 0 \\ q_{21} & q_{22} & 0 & \cdots & 0 \\ q_{31} & q_{32} & q_{33} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{n,n-1} & q_{nn} \end{pmatrix} \quad \text{and} \quad B = diag(0, 0, \ldots, 0, 1), \tag{2.7}$$

where

$$q_{ij} \geq q_0 > 0 \quad \text{in an open set } \mathcal{O}_0 \subset \mathcal{O}, \quad \forall i > j,\, i, j = 1, \ldots, n. \tag{2.8}$$

The following result holds from [28].

Lemma 2. *Assume that $Q^* \in L^\infty(\Omega)^{n^2}$ and $B \in \mathcal{M}_n(\mathbb{R})$ are given by* (2.7) *and satisfy* (2.8). *Let $\tau \in (0, T]$ and $\Xi_0 \in L^2(\Omega)^n$. Then, there exists a control function $U^{(\tau)} = U^{(\tau)}(\Xi_0) \in L^2(0, T; L^2(\mathcal{O})^n)$ such that the solution Ψ of the problem*

$$\begin{cases} -\partial_t \Psi - \Delta \Psi + Q^* \Psi = 1_{\mathcal{O}} B U^{(\tau)} & in \quad \Omega \times (0, \tau), \\ \Psi = 0 & on \quad \partial\Omega \times (0, \tau), \\ \Psi(\cdot, \tau) = \Xi_0 & in \quad \Omega, \end{cases} \tag{2.9}$$

satisfies

$$\Psi(\cdot, 0) = 0 \quad in\ \Omega. \tag{2.10}$$

Moreover, there exists a positive constant C_0 depending only on Ω and $\mathcal{O}$ such that

$$\|u_n^{(\tau)}\|_{L^2(0,T;L^2(\mathcal{O}))} \leq C_0 e^{C(\tau)} \|\Xi_0\|_{L^2(\Omega)^n}, \tag{2.11}$$

where

$$C(\tau) = \tau + \frac{1}{\tau} + \max_{j \leq i} \Big(|q_{ij}|^{2/(3(i-j)+3)} + \tau |q_{ij}| \Big).$$

Remark 2. *It is worth mentioning that Lemma 2 was proved in* [28, Theorem 1.2] *for a forward system instead of a backward system, however, the above configuration is more appropriate for solving our inverse source problem, Problem 1. In addition, we also mention that the null controllability property with one scalar control for general complete matrices is not possible* [28].

Concerning the system (1.2), its associated null controllability problem reads: given an initial datum $\Xi_0 \in L^2(\Omega)^n$, we look for a control function $U \in L^2(0, T; L^2(\mathcal{O})^n)$ such that the corresponding solution Ψ to

$$\begin{cases} -\partial_t \Psi - D^* \Delta \Psi = 1_{\mathcal{O}} B U & \text{in} \quad \Omega \times (0, T), \\ \Psi = 0 & \text{on} \quad \partial\Omega \times (0, T), \\ \Psi(\cdot, T) = \Xi_0 & \text{in} \quad \Omega, \end{cases} \tag{2.12}$$

satisfies the identity (2.10).

From [5, Remark 20], the system (2.12) can be controlled to zero with only one scalar control ($m = 1$) under the following assumptions (sufficient and necessary conditions):

A1) The diffusion matrix $D^*$is diagonalizable with positive real eigenvalues, i.e., for $J = diag(d_i)_{n\times n}$ with $d_1, d_2, \dots, d_n > 0$, one has $D^* = P^{-1}JP$, with $P \in \mathcal{M}_n(\mathbb{R}), detP \neq 0$.

A2) $d_i \neq d_j$, for $i \neq j,\ 1 \leq i, j \leq n$.

A3) $B = (b_1, \dots, b_n)^* \in \mathbb{R}^n$ and $b_i \neq 0$, for $i = 1, \dots, n$.

This information is summarized in the following Lemma 3.

Lemma 3. *For any $\Xi_0 \in L^2(\Omega)^n$, system* (2.12) *is null controllable with one scalar control if and only if the matrices D^* and B satisfy (A1)–(A3).*

Remark 3. *It is well known that the null controllability property for linear systems is equivalent to an observability inequality for the associated adjoint system. In relation to Lemma 3,* [4, page 271] *proves such an observability inequality through Carleman estimates. In the setting of inverse problems, stability aspects (Lipschitz–type) can be carried out using this comment, see remark 7 for additional information.*

2.3. **Volterra equations.** In this paragraph we recall technical results concerning scalar Volterra equations of first and second kind that we will need later on. We invite readers to see [27] and references therein for more details.

Lemma 4. *Assume $0 < t < \tau < T$, $\sigma \in W^{1,\infty}(0, \tau)$ and $\eta \in L^2(0, \tau; L^2(\Omega))$. Then, there exists a unique $\theta \in H^1(0, \tau; L^2(\Omega))$ satisfying the Volterra equation*

$$\begin{aligned} &\sigma(0)\partial_t\theta(x,t) + \int_t^\tau (\sigma(s-t)\theta(x,s) + \partial_t\sigma(s-t)\partial_t\theta(x,s))ds = \eta(x,t), \\ &\theta(x,\tau) = 0. \end{aligned} \tag{2.13}$$

Furthermore, there exists a constant $C > 0$ depending on $\|\sigma\|_{W^{1,\infty}(0,\tau)}$ such that

$$\|\theta\|_{H^1(0,\tau;L^2(\Omega))} \leq C\|\eta\|_{L^2(0,\tau;L^2(\Omega))}. \tag{2.14}$$

Now, let us consider the operator $K : L^2(0, T; L^2(\Omega)) \to H^1(0, T; L^2(\Omega))$ defined by

$$(Kv)(x,t) := \int_0^t \sigma(s)v(x,t-s)ds. \tag{2.15}$$

The following Lemma provides properties related to the operators K and K^*.

Lemma 5. *Assume $\sigma \in W^{1,\infty}(0, T)$. Then, there exist positive constants C_1 and C_2 depending only on Ω, T and $\|\sigma\|_{W^{1,\infty}(0,T)}$ such that*

$$C_1\|Kv\|_{H^1(0,T;L^2(\Omega))} \leq \|v\|_{L^2(0,T;L^2(\Omega))} \leq C_2\|Kv\|_{H^1(0,T;L^2(\Omega))}. \tag{2.16}$$

Furthermore, the adjoint operator $K^* : H^1(0, T; L^2(\Omega)) \to L^2(0, T; L^2(\Omega))$ is given by

$$(K^*\theta)(x,t) = \sigma(0)\partial_t\theta(x,t) + \int_t^T (\sigma(s-t)\theta(x,s) + \partial_t\sigma(s-t)\partial_t\theta(x,t))ds. \tag{2.17}$$

3. Systems with constant coefficients

In this section we present our first two main results associated to Problems 1–2 for the cases where the coupling matrices are constants. To do that, let us consider the systems

$$\begin{cases} \partial_t Y - \Delta Y + QY = \sigma(t)F(x) & \text{in} \quad \Omega \times (0,T), \\ Y = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ Y(\cdot,0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{3.1}$$

and

$$\begin{cases} \partial_t Y - D\Delta Y = \sigma(t)F(x) & \text{in} \quad \Omega \times (0,T), \\ Y = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ Y(\cdot,0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{3.2}$$

where $Q \in L^\infty(\Omega)^{n^2}$ satisfies (2.7), (2.8) and D satisfies the assumptions (A1) and (A2).

Let us observe that, thanks to the assumptions on the diffusion matrix D and the potential matrix Q, for every source $\sigma(t)F(x) \in L^2(0,T;L^2(\Omega)^n)$, system (3.1) (resp. system (3.2)) admits a unique weak solution $Y \in C([0,T];L^2(\Omega)^n) \cap L^2(0,T;H_0^1(\Omega)^n)$. Additionally, by considering $\sigma \in W^{1,\infty}(0,T)$ and following, for example, the procedure given in [32, Chapter 3, Section 6], solvability in the space $W_2^{2,1}(\Omega \times (0,T)) := L^2(0,T;H^2(\Omega)^n \cap H_0^1(\Omega)^n) \cap H^1(0,T;L^2(\Omega)^n)$ also holds for the systems (3.1) and (3.2).

Remark 4. *Before presenting the main results of this section, a common denominator to the systems* (3.1) *and* (3.2) *is the integral representation of their solutions, which are possible thanks to the linearity of those systems. More precisely, from the Duhamel principle, the solution Y from either of those systems can be written by*

$$Y(x,t) = \int_0^t \sigma(s)W(x,t-s)ds, \quad (x,t) \in \Omega \times (0,T), \tag{3.3}$$

where W satisfies

$$\begin{cases} \partial_t W - \Delta W + QW = 0 & \textit{in} \quad \Omega \times (0,T), \\ W = 0 & \textit{on} \quad \partial\Omega \times (0,T), \\ W(\cdot,0) = \sigma(0)F(\cdot) & \textit{in} \quad \Omega. \end{cases} \quad \textit{or} \quad \begin{cases} \partial_t W - D\Delta W = 0 & \textit{in} \quad \Omega \times (0,T), \\ W = 0 & \textit{on} \quad \partial\Omega \times (0,T), \\ W(\cdot,0) = \sigma(0)F(\cdot) & \textit{in} \quad \Omega. \end{cases} \tag{3.4}$$

Furthermore, since $\partial_t Y(x,t) = \sigma(0)W(x,t) + \int_0^t \partial_t\sigma(t-s)W(x,s)ds$, by evaluating at $t = T$ the main equations of (3.1) *and* (3.1)*, we obtain the following identity:*

$$\sigma(0)W(x,T) + \int_0^T \partial_t\sigma(T-s)W(x,s)ds - \Delta Y(x,T) + \int_0^T \sigma(s)QW(x,T-s)ds = \sigma(T)F(x) \tag{3.5}$$

and

$$\sigma(0)W(x,T) + \int_0^T \partial_t\sigma(T-s)W(x,s)ds - D\Delta Y(x,T) = \sigma(T)F(x). \tag{3.6}$$

As mentioned, our inverse source problems are linked to null controllability properties for the adjoint systems associated to (3.1) and (3.2), as well as spectral properties of the main operators. Thus, the rest of this section is dedicated to connect the identities (3.5) and (3.6) with those topics, and therefore, two relevant issues must be agreed. First, for the coupling matrices Q, $D \in \mathcal{M}_n(\mathbb{R})$ of (3.1) and (3.2), respectively, their adjoint matrices Q^*, $D^* \in \mathcal{M}_n(\mathbb{R})$ must satisfy the controllability constrains (see (2.8) and assumptions (A1)–(A3)). Second, respect to the spectral part, since in this case the entries of Q and D are constants, it is enough to consider the L^2–eigenfunctions and eigenvalues of the Laplace operator in Ω with Dirichlet boundary conditions. It will be denoted by $\{\varphi_k\}_{k\in\mathbb{N}}$ and $\{\lambda_k\}_{k\in\mathbb{N}}$ respectively. Finally, in order to present our first result, some additional hypotheses must be considered.

H1. *Consider $\sigma \in W^{1,\infty}(0,T)$ with $\sigma(T)\neq 0$. Furthermore, for some $k\in\mathbb{N}$*

$$a^{Q}_{j,k}(T) := \left(1-\frac{\lambda_k}{\sigma(T)}\sum_{i=1}^{n} m_{ij}(T)\right)\neq 0, \quad \forall i,j=1,\dots,n, \tag{3.7}$$

where $M=(m_{ij}(t))=\int_0^t \tilde{\Phi}_k(t)\tilde{\Phi}_k^{-1}(s)\sigma(s)ds$ and $\tilde{\Phi}_k$ is a fundamental matrix associated to the linear ordinary differential system: $Z'+(\lambda_k I_n+Q)Z=0$.

H2. *Consider Lemma 2, where $U^{(\tau)}=(u_1^{(\tau)},\dots,u_n^{(\tau)})^*$ is the control function associated to the system* (2.9) *extended by zero in (τ,T), and*

$$BU^{(\tau)}=(0,0,\dots,u_n^{(\tau)})^* \quad \textit{and } u_n^{(\tau)}\neq 0. \tag{3.8}$$

H3. *From H2, let $\Theta^{(\tau)}:=(\theta_1^{(\tau)},\dots,\theta_n^{(\tau)})$ be a solution to n copies of* (2.13) *with right–hand side given by* (3.8)*, i.e., (see Lemma 4 and Lemma 5)*

$$(K^*\Theta^{(\tau)})(x,t)=(\eta_1,\dots,\eta_n)=BU^{(\tau)}. \tag{3.9}$$

Our first source reconstruction result is given in the following theorem.

Theorem 1. *Let H1–H3 be satisfied. Then, for every solution $Y\in W_2^{2,1}(\Omega\times(0,T))$ to* (3.1)*, the source $F=(f_1,\dots,f_n)^*\in L^2(\Omega)^n$ satisfies the local reconstruction identity*

$$\begin{aligned}\sum_{j=1}^{n} a^{Q}_{j,k}(T)(f_j,\varphi_k)_{L^2(\Omega)} = &-\frac{\sigma(0)}{\sigma(T)}(y_n,(\theta_{1,k}^{(T)})_n)_{H^1(0,T;L^2(\mathcal{O}))}-\frac{1}{\sigma(T)}\int_0^T \partial_t\sigma(T-s)(y_n,(\theta_{1,k}^{(s)})_n)_{H^1(0,T;L^2(\mathcal{O}))}ds\\ &-\frac{1}{\sigma(T)}\int_0^T \sigma(T-s)(y_n,(\theta_{2,k}^{(s)})_n)_{H^1(0,T;L^2(\mathcal{O}))}ds.\end{aligned} \tag{3.10}$$

Proof. Due to that the proof essentially combines three different topics, we divide its proof in three steps. *Step 1. Spectral representation.* First, from the L^2–eigenfunctions $\{\varphi_k\}_{k\in\mathbb{N}}$ of the Laplace operator with homogeneous Dirichlet conditions, note that the solution of (3.1) can also be written as

$$Y(x,t)=\sum_{k\in\mathbb{N}} Y_k(t)\varphi_k(x), \tag{3.11}$$

where $Y_k(t)=(y_1^k(t),\dots,y_n^k(t))^*$ is the unique solution of the ordinary differential system

$$\left\{\begin{array}{l} Y_k'(t)+(\lambda_k I_n+Q)Y_k(t)=\sigma(t)F_k,\\ Y_k(0)=0,\end{array}\right. \tag{3.12}$$

where $F_k=((f_1,\varphi_k)_{L^2(\Omega)},\dots,(f_n,\varphi_k)_{L^2(\Omega)})^* =: (f_1^k,\dots,f_n^k)^*$.

By solving (3.12), for every $k \in \mathbb{N}$, we obtain

$$Y_k(t) = \overbrace{\left(\int_0^t \tilde{\Phi}_k(t) \tilde{\Phi}_k^{-1}(s) \sigma(s) ds \right)}^{M=(m_{ij}(t))_{i,j=1}^n} F_k = \left(\sum_{j=1}^n m_{1j}(t) f_j^k,\ \sum_{j=1}^n m_{2j}(t) f_j^k,\ \ldots, \sum_{j=1}^n m_{nj}(t) f_j^k \right)^*, \tag{3.13}$$

where $\tilde{\Phi}_k$ was defined in (3.7) (see assumption H1).

Now, let us define the vector $\Xi_k := (\varphi_k, \ldots, \varphi_k) \in L^2(\Omega)^n$ and consider the sequence $\mathcal{B} = \{\Xi_k\}_{k\in\mathbb{N}}$. By multiplying the identity (3.5) by elements of $\mathcal{B}$ and integrating in space, we get

$$\begin{aligned} \sigma(T)(F, \Xi_k)_{L^2(\Omega)^n} =& \sigma(0)(W(T), \Xi_k)_{L^2(\Omega)^n} + \int_0^T \partial_t \sigma(T-s)(W(s), \Xi_k)_{L^2(\Omega)^n} ds \\ & - (\Delta Y(T), \Xi_k)_{L^2(\Omega)^n} + \int_0^T \sigma(T-s)(QW(s), \Xi_k)_{L^2(\Omega)^n} ds. \end{aligned} \tag{3.14}$$

Using (3.11) and the fact that $\{\varphi_k\}_{k\in\mathbb{N}}$ are eigenfunctions of the Laplace operator, the third term in the right–hand side of (3.14) can be transformed as follows:

$$-(\Delta Y(T), \Xi_k)_{L^2(\Omega)^n} = -(Y(T), \Delta\Xi_k)_{L^2(\Omega)^n} = \lambda_k (Y(T), \Xi_k)_{L^2(\Omega)^n} = \lambda_k \sum_{j=1}^n y_j^k(T). \tag{3.15}$$

Thus, at this moment our reconstruction formula is given by

$$\begin{aligned} (F, \Xi_k)_{L^2(\Omega)^n} =& \frac{\sigma(0)}{\sigma(T)}(W(T), \Xi_k)_{L^2(\Omega)^n} + \frac{1}{\sigma(T)} \int_0^T \partial_t \sigma(T-s)(W(s), \Xi_k)_{L^2(\Omega)^n} ds \\ & + \frac{\lambda_k}{\sigma(T)} \sum_{j=1}^n y_j^k(T) + \frac{1}{\sigma(T)} \int_0^T \sigma(T-s)(QW(s), \Xi_k)_{L^2(\Omega)^n} ds, \end{aligned}$$

which is equivalent to (after taking into account (3.13)):

$$\begin{aligned} \sum_{j=1}^n \left(1 - \frac{\lambda_k}{\sigma(T)} \sum_{i=1}^n m_{ij}(T) \right) f_j^k =& \frac{\sigma(0)}{\sigma(T)}(W(T), \Xi_k)_{L^2(\Omega)^n} + \frac{1}{\sigma(T)} \int_0^T \partial_t \sigma(T-s)(W(s), \Xi_k)_{L^2(\Omega)^n} ds \\ & + \frac{1}{\sigma(T)} \int_0^T \sigma(T-s)(QW(s), \Xi_k)_{L^2(\Omega)^n} ds. \end{aligned} \tag{3.16}$$

Step 2. Controllability. Now, in order to replace the global terms $(W(s), \Xi_k)_{L^2(\Omega)^n}$ and $(QW(s), \Xi_k)_{L^2(\Omega)^n}$ by local terms in $L^2(0, s; L^2(\mathcal{O})^n)$, for every $s \in (0, T]$, we use the null controllability property for adjoint systems associated to (3.1). In other words, we apply the hypothesis H2 for every $k \in \mathbb{N}$. Thus, if Ψ denotes the adjoint state of Y and $\overline{\Psi} := Q^*\Psi$, Lemma 2 guarantees the existence of a control

$U^{(s)} = U^{(s)}(\Xi_k) =: U_k^{(s)} \in L^2(0,s;L^2(\mathcal{O})^n)$ such that the systems

$$\begin{cases} -\partial_t \Psi - \Delta\Psi + Q^*\Psi = 1_{\mathcal{O}} B U_k^{(s)} & \text{in} \quad \Omega\times(0,s), \\ \Psi = 0 & \text{on} \quad \partial\Omega\times(0,s), \\ \Psi(\cdot,s) = \Xi_k(\cdot) & \text{in} \quad \Omega \end{cases} \tag{3.17}$$

and

$$\begin{cases} -\partial_t \overline{\Psi} - \Delta\overline{\Psi} + Q^*\overline{\Psi} = 1_{\mathcal{O}} Q^* B U_k^{(s)} & \text{in} \quad \Omega\times(0,s), \\ \Psi = 0 & \text{on} \quad \partial\Omega\times(0,s), \\ \overline{\Psi}(\cdot,s) = Q^*\Xi_k(\cdot) & \text{in} \quad \Omega, \end{cases} \tag{3.18}$$

satisfy

$$\Psi(x,0) = \overline{\Psi}(x,0) = 0, \quad \forall x\in\Omega. \tag{3.19}$$

Additionally, by multiplying the first system of (3.4) by Ψ solution of (3.17) and (3.19), and integrating by parts in $L^2(0,s;L^2(\Omega)^n)$, we obtain (after extending $U_k^{(s)}$ by zero at (s,T))

$$(W(s),\Xi_k)_{L^2(\Omega)^n} = -(W, 1_{\mathcal{O}} B U_k^{(s)})_{L^2(0,s;L^2(\Omega)^n)} = -(W, BU_k^{(s)})_{L^2(0,T;L^2(\mathcal{O})^n)}. \tag{3.20}$$

Analogously, for $\overline{\Psi}$ solution of (3.18) and (3.19) we have

$$(QW(s),\Xi_k)_{L^2(\Omega)^n} = -(W, 1_{\mathcal{O}} Q^* B U_k^{(s)})_{L^2(0,s;L^2(\Omega)^n)} = -(W, Q^* BU_k^{(s)})_{L^2(0,T;L^2(\mathcal{O})^n)}. \tag{3.21}$$

Step 3. Volterra equations. In this step, we essentially adapt subsection 2.3 to the case of systems of Volterra equations and we make the relation (3.9) established in H3 in order to ensure an appropriate connexion with (3.20) and (3.21).

First, we apply twice Lemma 4 in a vector form with data $\eta_k^1 := 1_{\mathcal{O}} BU_k^{(s)}$, and also with data $\eta_k^2 := 1_{\mathcal{O}} Q^* BU_k^{(s)}$. After that, Lemma 5 guarantees the identities

$$K^*(\Theta_k^1) = 1_{\mathcal{O}} BU_k^{(s)} \quad \text{and} \quad K^*(\Theta_k^2) = 1_{\mathcal{O}} Q^* BU_k^{(s)}, \quad \forall k\in\mathbb{N}.$$

Then, replacing the above relations in (3.20) and (3.21), we obtain

$$(W(s),\Xi_k)_{L^2(\Omega)^n} = -(W, BU_k^{(s)})_{L^2(0,T;L^2(\mathcal{O})^n)} = -(W, K^*(\Theta_k^1))_{L^2(0,T;L^2(\mathcal{O})^n)},$$

$$(QW(s),\Xi_k)_{L^2(\Omega)^n} = -(W, Q^*BU_k^{(s)})_{L^2(0,T;L^2(\mathcal{O})^n)} = -(W, K^*(\Theta_k^2))_{L^2(0,T;L^2(\mathcal{O})^n)}.$$

Note that, thanks to (3.9) and (3.8), we really have

$$\begin{aligned} (W(s),\Xi_k)_{L^2(\Omega)^n} &= -(W, K^*(\Theta_k^1))_{L^2(0,T;L^2(\mathcal{O})^n)} = -(w_n, K^*(\theta_k^1)_n)_{L^2(0,T;L^2(\mathcal{O}))} \\ (QW(s),\Xi_k)_{L^2(\Omega)^n} &= -(W, K^*(\Theta_k^2))_{L^2(0,T;L^2(\mathcal{O})^n)} = -(w_n, K^*(\theta_k^2)_n)_{L^2(0,T;L^2(\mathcal{O}))}. \end{aligned} \tag{3.22}$$

From (2.15), it is easy to deduce that $Y = KW$, in particular $y_n = Kw_n$. Therefore, it follows

$$(w_n, K^*(\theta_k^j)_n)_{L^2(0,T;L^2(\mathcal{O}))} = (y_n, (\theta_k^j)_n)_{H^1(0,T;L^2(\mathcal{O}))}, \quad j=1,2. \tag{3.23}$$

Finally, putting together (3.16), (3.22) and (3.23) we obtain the desired reconstruction formula (3.10), which completes the proof of Theorem 1. □

Remark 5. *Under the hypothesis H1, the reconstruction formula* (3.10) *holds for every $k\in\mathbb{N}$ in the following particular cases of time dependency σ of the source (see* [27, 26]*):*

a) $\sigma(t) = \sigma_0$ *constant.*
b) σ *a non–negative and increasing function.*
b) $\sigma(t) = 1 + \frac{1}{2}\cos\left(\frac{4t}{T-\varepsilon}\right)$ *for $t < T-\varepsilon$, and $\sigma(t) = \frac{3}{2}$ for $t > T-\varepsilon$.*

Now, we establish the necessary hypothesis for solving Problem 2, which in our case is related to the system (3.2).

H4. *Consider $\sigma \in W^{1,\infty}(0,T)$ with $\sigma(T) \neq 0$. Furthermore, for some $k \in \mathbb{N}$*

$$a_{j,k}^D(T) := \left(1 - \frac{\lambda_k}{\sigma(T)} \sum_{\ell=1}^n \left(\sum_{i=1}^n d_{i\ell}\right) m_{\ell j}(T)\right) \neq 0, \quad \forall i,j = 1,\dots,n, \tag{3.24}$$

where $M = (m_{ij}(t)) = \int_0^t \tilde{\Phi}_k(t)\tilde{\Phi}_k^{-1}(s)\sigma(s)ds$ and $\tilde{\Phi}_k$ a fundamental matrix associated to the ordinary differential system: $Z' + \lambda_k DZ = 0$.

H5. *Assume Lemma 3.*

H6. *From H5, let $\Theta^{(\tau)} := (\theta_1^{(\tau)},\dots,\theta_n^{(\tau)})$ be a solution to n copies of* (2.13) *with right–hand side given by $BU^{(\tau)}$, where $B \in \mathbb{R}^n$ satisfy the condition (A3). That is, we consider the following identity (see* (2.13) *and Lemma 2.17):*

$$(K^*\Theta^{(\tau)})(x,t) = (\eta_1,\dots,\eta_n) = BU^{(\tau)}. \tag{3.25}$$

Our second source reconstruction result is given in the following theorem.

Theorem 2. *Let H4–H6 be satisfied. Then, for every solution $Y \in W_2^{2,1}(\Omega\times(0,T))$ to* (3.1)*, the source $F = (f_1,\dots,f_n)^* \in L^2(\Omega)^n$ satisfies the local reconstruction identity*

$$\sum_{j=1}^n a_{j,k}^D(T)(f_j,\varphi_k)_{L^2(\Omega)} = -\frac{\sigma(0)}{\sigma(T)}(y_n,(\theta_k)_n)_{H^1(0,T;L^2(\mathcal{O}))} - \frac{1}{\sigma(T)}\int_0^T \partial_t\sigma(T-s)(y_n,(\theta_k)_n)_{H^1(0,T;L^2(\mathcal{O}))}ds. \tag{3.26}$$

Proof. Following the structure of the proof of Theorem 1, we have again three steps.
Step 1. Spectral representation. First, we consider the representation to the solution of (3.2) in a serie of the form (3.11), where now $Y_k(t) = (y_1^k(t),\dots,y_n^k(t))^*$ is the unique solution of the ordinary differential system

$$\begin{cases} Y_k'(t) + \lambda_k DY_k(t) = \sigma(t)F_k, \\ Y_k(0) = 0, \end{cases} \tag{3.27}$$

where $F_k = ((f_1,\varphi_k)_{L^2(\Omega)},\dots,(f_n,\varphi_k)_{L^2(\Omega)})^* =: (f_1^k,\dots,f_n^k)^*$.

Now, by solving (3.27) we obtain

$$Y_k(t) = \overbrace{\left(\int_0^t \tilde{\Phi}_k(t)\tilde{\Phi}_k^{-1}(s)\sigma(s)ds\right)}^{M=(m_{ij}(t))} F_k = \left(\sum_{j=1}^n m_{1j}(t)f_j^k,\ \sum_{j=1}^n m_{2j}(t)f_j^k,\ \dots, \sum_{j=1}^n m_{nj}(t)f_j^k\right)^* \tag{3.28}$$

where $\tilde{\Phi}_k$ was defined in (3.24) (see assumption H4).

Again, we consider the vector $\Xi_k := (\varphi_k,\dots,\varphi_k) \in L^2(\Omega)^n$ and the sequence $\mathcal{B} = \{\Xi_k\}_{k\in\mathbb{N}}$. In addition, we consider the integral representation (3.3), where now W satisfies the system located in the right–hand side of (3.4). Thus, by multiplying the identity (3.6) by elements of $\mathcal{B}$ and integrating in space, we get

$$\begin{aligned}\sigma(T)(F,\Xi_k)_{L^2(\Omega)^n} =& \sigma(0)(W(T),\Xi_k)_{L^2(\Omega)^n} + \int_0^T \partial_t\sigma(T-s)(W(s),\Xi_k)_{L^2(\Omega)^n}ds \\ &- (D\Delta Y(T),\Xi_k)_{L^2(\Omega)^n}.\end{aligned}$$

Now, the last term in the above identity can be transformed as follows:

$$-(D\Delta Y(T),\Xi_k)_{L^2(\Omega)^n}=-(DY(T),\Delta\Xi_k)_{L^2(\Omega)^n}=\lambda_k\sum_{j=1}^n\sum_{i=1}^n d_{ij}y_j^k(T),\quad \forall k\in\mathbb{N}.$$

Then, at this moment our reconstruction formula is given by

$$\begin{aligned}(F,\Xi_k)_{L^2(\Omega)^n}=&\frac{\sigma(0)}{\sigma(T)}(W(T),\Xi_k)_{L^2(\Omega)^n}+\frac{1}{\sigma(T)}\int_0^T\partial_t\sigma(T-s)(W(s),\Xi_k)_{L^2(\Omega)^n}ds\\&+\frac{\lambda_k}{\sigma(T)}\sum_{j=1}^n\sum_{i=1}^n d_{ij}y_j^k(T),\end{aligned}$$

or equivalently (taking into account (3.28))

$$\begin{aligned}\sum_{j=1}^n\left(1-\frac{\lambda_k}{\sigma(T)}\sum_{\ell=1}^n\left(\sum_{i=1}^n d_{i\ell}\right)m_{\ell j}(T)\right)f_j^k=&\frac{\sigma(0)}{\sigma(T)}(W(T),\Xi_k)_{L^2(\Omega)^n}\\&+\frac{1}{\sigma(T)}\int_0^T\partial_t\sigma(T-s)(W(s),\Xi_k)_{L^2(\Omega)^n}ds.\end{aligned}\tag{3.29}$$

Step 2. Controllability. In order to transform the global terms of (3.29) by local terms in the subdomain $\mathcal{O}\times(0,T)$, we apply the null controllability property for parabolic systems whose coupling is in the main operator, see Lemma 3. Then, if Ψ denotes the adjoint state of Y, Lemma 3 allow to obtain a control function $U^{(s)}=U^{(s)}(\Xi_k)=:U_k^{(s)}\in L^2(0,s;L^2(\mathcal{O})^n)$ such that the system

$$\begin{cases}-\partial_t\Psi-D^*\Delta\Psi=1_{\mathcal{O}}BU_k^{(s)} & \text{in}\quad \Omega\times(0,s),\\ \Psi=0 & \text{on}\quad \partial\Omega\times(0,s),\\ \Psi(\cdot,s)=\Xi_s(\cdot) & \text{in}\quad \Omega\end{cases}\tag{3.30}$$

satisfy

$$\Psi(x,0)=0,\quad \forall x\in\Omega.\tag{3.31}$$

Thus, by multiplying the second system of (3.4) by Ψ solution of (3.30) and (3.31), and integrating by parts in $L^2(0,s;L^2(\Omega)^n)$, we obtain (after extending $U_k^{(s)}$ by zero at (s,T))

$$(W(s),\Xi_k)_{L^2(\Omega)^n}=-(W,1_{\mathcal{O}}BU_k^{(s)})_{L^2(0,s;L^2(\Omega)^n))}=-(W,BU_k^{(s)})_{L^2(0,T;L^2(\Omega)^n))}.\tag{3.32}$$

It worth mentioning that the above identity is possible thanks to the fact that D is a constant matrix and the operator $A=-\Delta$ is self–adjoint, since otherwise the analysis might be rather complex.

Step 3. Volterra equations. Following the arguments of step 3 of the proof of Theorem 1, we use the relation (3.25) given in assumption H6. More precisely, from (3.25), Lemma 4 and Lemma 5 we can deduce

$$(W(s),\Xi_k)_{L^2(\Omega)^n}=-(W,BU_k^{(s)})_{L^2(0,T;L^2(\Omega)^n))}=-(w_n,K^*(\theta_k)_n)_{H^1(0,T;L^2(\mathcal{O}))}.\tag{3.33}$$

Finally, from (2.15) we have $y_n=Kw_n$, and in consequence

$$-(w_n,K^*(\theta_k)_n)_{H^1(0,T;L^2(\mathcal{O}))}=-(y_n,(\theta_k)_n)_{H^1(0,T;L^2(\mathcal{O}))}.\tag{3.34}$$

Therefore, putting together (3.29), (3.33) and (3.34) we establish the formula (3.26). This completes the proof of Theorem 2. □

4. Systems with space dependent coefficients

In this section, we are interested in the question of recovering the spatial dependence of a source for the following coupled system of second–order parabolic equations

$$\begin{cases} \partial_t Y + (\overbrace{-\Delta + Q(x)}^{L})Y = \sigma(t)F(x) & \text{in } (0,\pi)\times(0,T), \\ Y(0,t) = Y(\pi,t) = 0 & \text{in } (0,T), \\ Y(\cdot,0) = 0 & \text{in } (0,\pi), \end{cases} \tag{4.1}$$

where $L : H^2(0,\pi)^2 \cap H_0^1(0,\pi)^2 \subset L^2(0,\pi)^2 \to L^2(0,\pi)^2$ and Q is given by

$$Q(x) = \begin{pmatrix} 0 & 0 \\ q(x) & 0 \end{pmatrix} \quad \text{and} \quad q \in L^\infty(0,\pi) \cap W^{1,\infty}(\tilde{\mathcal{O}}),\ \tilde{\mathcal{O}} \subset \mathcal{O} \subset (0,\pi).$$

In contrast to the previous section where the corresponding coupling terms are constants and the symmetry of the self–adjoint operator $A = -\Delta$ allow to make auxiliar control systems for estimating global terms in an easy way (see step 2 inside the proof of Theorem 1 and Theorem 2), the strategy for recovering the source $F(x)$ of system (4.1) is very different. It mainly relies on three issues: a) properties on the eigenfunctions associated with non–self–adjoint matrix Sturm–Liouville operators, namely, the operators $L := -\Delta + Q(x)$ and $L^* := -\Delta + Q(x)^*$; b) a null controllability result related to the adjoint system of (4.1). The above regularity on the coefficient q is used in this part (see [21, theorem 1.1]); and again, c) Volterra equations.

On the other hand, as mentioned in Remark 1, the spectral study for a system of $n \times n$ equations with matrix operator L is more difficult to investigate, involving concepts that are not considered in this paper. Furthermore, for a higher space dimension, an exhaustive revision from spectral theory [30] might be useful in order to obtain a source reconstruction formula as in theorem below for more general systems to the presented in (4.1).

Before presenting the main theorem of this section, we will first consider some hypotheses.

H7. *Consider $\sigma \in W^{1,\infty}(0,T)$ with $\sigma(T) \neq 0$. Furthermore, for some $k \in \mathbb{N}$*

$$a_k^L(T) := \sigma(T)\left(1 - \frac{k^2}{\sigma(T)}\int_0^T e^{-k^2(T-s)}\sigma(s)ds\right) \neq 0,\ b_k^L(T) := -I_k(q)\Big(1 - k^2\int_0^T (T-s)e^{-k^2(T-s)}\sigma(s)ds\Big). \tag{4.2}$$

H8. *Consider Lemma 1.*

H 9. *For any $s \in (0,T]$, assume that the adjoint system associated to* (4.1) *with distributed control $1_{\mathcal{O}}U^{(s)} = (0, 1_{\mathcal{O}}u_2^{(s)})$ satisfies the null controllability property (see* [21]*).*

H10. *Consider Lemmas 4 and 5.*

Theorem 3. *Let H7–H10 be satisfied. The, for any solution $Y \in W_2^{2,1}((0,\pi)\times(0,T))$ of* (4.1)*, the source $F = (f_1, f_2) \in L^2(0,\pi)^2$ satisfies*

$$a_k^L(T)\Big(f_1^{\varphi_k} + f_1^{\psi_k} + f_2^{\varphi_k}\Big) + b_k^L(T)f_1^{\varphi_k} = -\sigma(0)(y_2, \theta_k^{(s)})_{H^1(0,T;L^2(\mathcal{O}))} - \int_0^T \partial_t\sigma(T-s)(y_2,\theta_k^{(s)})_{H^1(0,T;L^2(\mathcal{O}))}ds, \tag{4.3}$$

where $f_1^{\varphi_k} := (f_1,\varphi_k)_{L^2(0,\pi)}$, $f_1^{\psi_k} := (f_1,\psi_k)_{L^2(0,\pi)}$ and $f_2^{\varphi_k} := (f_2,\varphi_k)_{L^2(0,\pi)}$.

Proof. We will follow the strategy of the above section. The main novelty relies on the spectral part for the operators L and L^*, and whose representation was given in subsection 2.1. Again, the proof is divided in three steps.

Step 1. Spectral property. The main ingredient in this step is the spectrum of the operators L and L^*. First, returning to (2.3), for every $k \in \mathbb{N}$, we consider the normalized eigenfunctions $\varphi_k(x) = \sqrt{\frac{2}{\pi}}\sin(kx)$ of the Laplace operator ∂_{xx} with Dirichlet boundary conditions over $(0,\pi)$.

Secondly, thanks to Lemma 1, the solution to (4.1) can be represented in the form:

$$Y(x,t) = \sum_{k\in\mathbb{N}} \alpha_k(t)\Phi_{1,k}(x) + \beta_k(t)\Phi_{2,k}(x), \tag{4.4}$$

where $\alpha_k(t) = (Y(t),\Phi_{1,k})_{L^2(0,\pi)^2}, \beta_k(t) = (Y(t),\Phi_{2,k})_{L^2(0,\pi)^2}$, and $\Phi_{1,k},\Phi_{2,k}$ belong to the family $\mathcal{B}$, which constitutes a Riesz basis of $L^2(0,\pi)^2$. Moreover, the sequences $\{\alpha_k(T)\}_{k\in\mathbb{N}}$ and $\{\beta_k(T)\}_{k\in\mathbb{N}}$ can be determined in explicit form, it is a consequence of the biorthogonal property between the families $\mathcal{B}$ and $\mathcal{B}^*$ as well as the relations (2.5) and (2.6). Thus, we can deduce the coupled system of ordinary differential equations

$$\frac{d}{dt}\begin{pmatrix}\alpha_k(t)\\ \beta_k(t)\end{pmatrix} + \begin{pmatrix}k^2 & I_k(q)\\ 0 & k^2\end{pmatrix}\begin{pmatrix}\alpha_k(t)\\ \beta_k(t)\end{pmatrix} = \sigma(t)\begin{pmatrix}(F,\Phi^*_{1,k})_{L^2(0,\pi)^2}\\ (F,\Phi^*_{2,k})_{L^2(0,\pi)^2}\end{pmatrix}, \quad \forall\, k\in\mathbb{N},$$

where $I_k(q) := \int_0^\pi q(x)\varphi_k(x)dx$.

From the definitions of $\Phi^*_{1,k}$ and $\Phi^*_{2,k}$ (see Lemma 1), the explicit solution to the previous system corresponds to:

$$\begin{cases}\beta_k(t) = f_1^{\varphi_k}\displaystyle\int_0^t e^{-k^2(t-s)}\sigma(s)ds.\\[2ex] \alpha_k(t) = (f_1^{\psi_k}+f_2^{\varphi_k})\displaystyle\int_0^t e^{-k^2(t-s)}\sigma(s)ds - I_k(q)\int_0^t e^{-k^2(t-s)}\beta_k(s)ds,\end{cases} \tag{4.5}$$

where by simplicity $f_1^{\varphi_k} := (f_1,\varphi_k)_{L^2(0,\pi)}$, $f_1^{\psi_k} := (f_1,\psi_k)_{L^2(0,\pi)}$ and $f_2^{\varphi_k} := (f_2,\varphi_k)_{L^2(0,\pi)}$.

On the other hand, note that the integral representation (3.3) holds, as well as a similar system to (3.4). More precisely, for every $(x,t)\in(0,\pi)\times(0,T)$ we have

$$Y(x,t) = \int_0^t \sigma(s)W(x,t-s)ds \quad\text{and}\quad \begin{cases}\partial_t W + LW = 0 & \text{in } (0,\pi)\times(0,T),\\ W(0,t) = W(\pi,t) = 0 & \text{in } (0,T),\\ W(\cdot,0) = \sigma(0)F(\cdot), & \text{in } (0,\pi).\end{cases} \tag{4.6}$$

Since $\partial_t Y(x,t) = \sigma(0)W(x,t) + \int_0^t \partial_t\sigma(t-s)W(x,s)ds$, then, by evaluating the main equations of (4.1) in $t=T$, and multiplying by elements of the family $\mathcal{B}^*$ and integrating over $(0,\pi)$, we get

$$\begin{aligned}(\sigma(T)F,\Phi^*_{1,k}+\Phi^*_{2,k})_{L^2(0,\pi)^2} =& \sigma(0)(W(T),\Phi^*_{1,k}+\Phi^*_{2,k})_{L^2(0,\pi)^2}\\ &+\int_0^T \partial_t\sigma(T-s)(W(s),\Phi^*_{1,k}+\Phi^*_{2,k})_{L^2(0,\pi)^2}ds\\ &+(LY(T),\Phi^*_{1,k}+\Phi^*_{2,k})_{L^2(0,\pi)^2}.\end{aligned} \tag{4.7}$$

Using again Lemma 1 and (4.4), the last term in the right–hand side of (4.7) can be transformed as follows:

$$\begin{aligned}(LY(T),\Phi^*_{1,k}+\Phi^*_{2,k})_{L^2(0,\pi)^2} =& (Y(T),I_k(q)\Phi^*_{2,k})_{L^2(0,\pi)^2} + k^2(\alpha_k(T)+\beta_k(T))\\ =& (I_k(q)+k^2)\beta_k(T) + k^2\alpha_k(T).\end{aligned} \tag{4.8}$$

Thus, at this moment our reconstruction formula is given by

$$\begin{aligned}\sigma(T)(F,\Phi^*_{1,k}+\Phi^*_{2,k})_{L^2(0,\pi)^2}=&\sigma(0)(W(T),\Phi^*_{1,k}+\Phi^*_{2,k})_{L^2(0,\pi)^2}\\&+\int_0^T\partial_t\sigma(T-s)(W(s),\Phi^*_{1,k}+\Phi^*_{2,k})_{L^2(0,\pi)^2}ds\\&+(I_k(q)+k^2)\beta_k(T)+k^2\alpha_k(T).\end{aligned}\tag{4.9}$$

where $\alpha_k(T)$ and $\beta_k(T)$ have been obtained in (4.5).
Step 2. Controllability. In relation to the proof of theorem 1 where the global terms are changed by using two adjoint systems and applying its respective controllability results, here, the global terms $(\mathbf{w}(\cdot,s),\Phi^*_{1,k}+\Phi^*_{2,k})_{L^2(0,\pi)^2}$ for $s\in(0,T]$ are replaced by local terms on $L^2(0,s;L^2(\mathcal{O}))^2$ by considering the hypothesis H9. More precisely, we consider the function $\Phi^*_{1,k}+\Phi^*_{2,k}$ as initial datum for the distributed control system

$$\begin{cases}-\partial_t\Psi+L^*\Psi=(0,1_{\mathcal{O}}u_2^{(s)}) & \text{in } (0,\pi)\times(0,s),\\ \Psi(0,t)=\Psi(L,t)=0 & \text{in } (0,s),\\ \Psi(\cdot,0)=\Xi_0(\cdot) & \text{in } (0,\pi),\end{cases}\tag{4.10}$$

where Ψ satisfies

$$\Psi(\cdot,0)=0\quad\text{in } (0,\pi).$$

Thus, raising as in the previous section, for every $s\in(0,T]$ follows

$$(W(\cdot,s),\Phi^*_{1,k}+\Phi^*_{2,k})_{L^2(0,\pi)^2}=-(W,(0,1_{\mathcal{O}}u_2^{(s)}))_{L^2(0,s;L^2(\mathcal{O})^2)}=-(w_2,u_2^{(s)})_{L^2(0,T;L^2(\mathcal{O}))}.\tag{4.11}$$

Step 3. Volterra equation. Let $\theta_k^{(s)}$ be a solution to (2.13) with right–hand side given by $1_{\mathcal{O}}u_2^{(s)}$. Using (2.13) and Lemma 5, we get $K^*\theta_k^{(s)}=1_{\mathcal{O}}u_2^{(s)}$. Replacing this into (4.11), we have

$$(W(\cdot,s),\Phi^*_{1,k}+\Phi^*_{2,k})_{L^2(0,\pi)^2}=-(w_2,u_2^{(s)})_{L^2(0,T;L^2(\mathcal{O}))}=-(w_2,K^*\theta_k^{(s)})_{L^2(0,T;L^2(\mathcal{O}))}.\tag{4.12}$$

Again, from (2.15), $y_2=Kw_2$ and in consequence

$$(W(\cdot,s),\Phi^*_{1,k}+\Phi^*_{2,k})_{L^2(0,\pi)^2}=-(w_2,K^*\theta_k^{(s)})_{L^2(0,T;L^2(\mathcal{O}))}=-(y_2,\theta_k^{(s)})_{H^1(0,T;L^2(\mathcal{O}))}\tag{4.13}$$

Finally, putting together (4.5), (4.9) and (4.13), our reconstruction formula is

$$a(T)(f_1^{\psi_k}+f_2^{\varphi_k}+f_1^{\varphi})+b(T)f_1^{\varphi}=-\sigma(0)(y_2,\theta_k^{(s)})_{H^1(0,T;L^2(\mathcal{O}))}-\int_0^T\partial_t\sigma(T-s)(y_2,\theta_k^{(s)})_{H^1(0,T;L^2(\mathcal{O}))}ds,$$

where

$$a(T)=\sigma(T)\left(1-\frac{k^2}{\sigma(T)}\int_0^T e^{-k^2(T-s)}\sigma(s)ds\right)$$

and

$$b(T)=-I_k(q)\left(1-k^2\int_0^T e^{-k^2(T-s)}\left(\int_0^s e^{-k^2(s-\tau)}\sigma(\tau)d\tau\right)ds\right).$$

This completes the proof of theorem 3. $\square$

Remark 6. *Note that, the advantage of using the linear combination $\Phi^*_{1,k}+\Phi^*_{2,k}$ in the above proof lies in the fact that we can recovery all coefficients for each term of $F=(f_1,\dots,f_n)^*$ through subspaces of $L^2(0,\pi)$, i.e., for the source $f_1\in L^2(0,\pi)$, we have $L^2=H_1\oplus H_2$, and it admits the representation $f_1=\sum_{k\in\mathbb{N}} f_1^{\varphi_k}\varphi_k+f_1^{\psi_k}\psi_k$, where $H_1=\langle\{\varphi_k : k\in\mathbb{N}\}\rangle$ and $H_2=\langle\{\psi_k : k\in\mathbb{N}\}\rangle$; meanwhile, since the coupling occurs in the second equation, for $f_2\in L^2(0,\pi)$, we only obtain $P_{H_1}f_2$, where P_{H_1} represents the orthogonal projector from $L^2(0,\pi)$ onto H_1. Nevertheless, H_1 constitutes an orthonormal basis of $L^2(0,\pi)$, and therefore the source reconstruction formula* (4.3) *shows every term of F in complete form.*

Remark 7. *In concordance with the scalar case* [27], *if $\partial_t\sigma(t)=0$ for $t\in(T-\varepsilon,T)$ for some $\varepsilon>0$ or, if $\partial_t\sigma(t)$ decreases exponentially in $(0,T)$, Lipschitz stability properties for the reconstruction formulas given in Theorems 1–3 also hold. It is easy to verify that from* [27, Step 4, page 764] *and therefore we have decided to omit it. Nevertheless, the logarithmical stability problem linked to a more regular source (i.e., $F\in(D(-\Delta^\varepsilon))^n$) and a reduced number of local interior observations remains open.*

5. Lipschitz stability

The main goal in this section is to give an answer to Problem 3. More precisely, we prove a stability estimate of Lipschitz type in determining the sources $f_1,\dots f_n$ of the system (1.3) by data of only one component. As mentioned in Section 1, the strategy is based in the Bukhgeim–Klibanov method [31]. As basic tool, we will use a global Carleman inequality satisfied by the solutions of

$$\begin{cases} \partial_t Y-\Delta Y+QY=H & \text{in} \quad \Omega\times(0,T),\\ Y=0 & \text{in} \quad \partial\Omega\times(0,T),\\ Y(\cdot,0)=0 & \text{in} \quad \Omega, \end{cases} \tag{5.1}$$

where $H=(h_1,\dots,h_n)^*\in L^2(0,T;L^2(\Omega)^n)$ and $Q\in L^\infty(\Omega)^{n^2}$ is the coupling matrix defined in (2.7) and (2.8). One has:

Lemma 6. *Let $\omega\subset\Omega$ be a nonempty open subset and $d\le 3$. Then, there exists a function $\alpha_0\in C^2(\overline{\Omega})$ (only depending on Ω and ω) and three positive constants $s_1=s_1(\Omega,\omega,T,\|q_{ij}\|_\infty)$, $C=C(\Omega,\omega,T,q_0)$ and $\ell>3$ such that, for any $s\ge s_1$, the solution Y to* (5.1) *satisfies*

$$\sum_{i=1}^n \mathcal{I}(3(n+1-i),y_i)\le C\left(s^\ell\iint\limits_{\omega\times(0,T)} e^{-2s\alpha}\xi(t)^\ell|y_n|^2dxdt+s^{d-3}\sum_{i=1}^n\iint\limits_{\Omega\times(0,T)} e^{-2s\alpha}\xi(t)^{d-3}|h_i|^2dxdt\right), \tag{5.2}$$

where the term $\mathcal{I}(d,z)$ is given by

$$\mathcal{I}(d,z)=\iint\limits_{\Omega\times(0,T)} e^{-2s\alpha}\left((s\xi)^{d-4}|\partial_t z|^2+(s\xi)^{d-2}|\nabla z|^2+(s\xi)^d|z|^2\right)dxdt, \tag{5.3}$$

and the functions α and ξ are given by

$$\alpha(x,t)=\frac{\alpha_0(x)}{t(T-t)},\ (x,t)\in\Omega\times(0,T),\quad \xi(t)=(t(T-t))^{-1}, t\in(0,T).$$

Remark 8. *The proof of Lemma 6 can be found in* [28, Theorem 1.1]. *It is worth mentioning that the first term of $\mathcal{I}(d,z)$ as well as the last term in the right–hand side of* (5.2) *does not appear into* [28, Theorem 1.1]. *However, by tracking their arguments is possible to add those terms in an easy way. Additionally, although Lemma 6 holds for $d\in\mathbb{R}$, our restriction to $d\le 3$ is required in order to prove the Lipschitz stability associated to our inverse source problem (Problem 3).*

Before mentioning the main result of this section, we impose several assumptions.

H11. *Consider $G \in W^{1,\infty}(\mathbb{R}^n)^n$ and $\partial_t G \in L^2(0,T;L^2(\Omega)^n)^n$.*

H 12. *Consider $\sigma \in W^{1,\infty}(0,T)$. Moreover, for $T' = \frac{T}{2}$, there exists a positive constant γ_0 such that $|\sigma(T')| \geq \gamma_0 > 0$.*

H13. *Let $T' = \frac{T}{2}$. Assume $Y(\cdot,T') = \tilde{Y}(\cdot,T')$ in Ω, where Y and $\tilde{Y}$ are solutions to* (1.3) *(with $A = -\Delta$) for sources $\sigma(t)F(x)$ and $\sigma(t)\tilde{F}(x)$, respectively.*

The main result of this section is the following theorem.

Theorem 4. *Let H11–H13 be satisfied. Then, there exists a positive constant $C = C(\Omega,\omega,T',T,q_0)$ such that*

$$\|F - \tilde{F}\|_{L^2(\Omega)^n} \leq C\|\partial_t y_n - \partial_t \tilde{y}_n\|_{L^2(0,T;L^2(\omega))}. \tag{5.4}$$

Proof. Let Y and $\tilde{Y}$ be solutions of (1.3) associated to the sources σF and $\sigma\tilde{F}$, respectively. If we define $U := Y - \tilde{Y}$ and $Z := \partial_t U$, then

$$\begin{cases} \partial_t Z - \Delta Z + QZ + G(Y) - G(\tilde{Y}) = (\partial_t \sigma)(F - \tilde{F}) & \text{in} \quad \Omega \times (0,T), \\ Z = 0 & \text{in} \quad \partial\Omega \times (0,T). \end{cases} \tag{5.5}$$

Using hypotheses H11, H12 and applying the Carleman estimate (5.2) to system (5.5), we obtain

$$\begin{aligned} \sum_{i=1}^{n} \mathcal{I}(3(n+1-i), z_i) \quad &\leq Cs^{\ell} \iint\limits_{\omega\times(0,T)} e^{-2s\alpha}\xi(t)^{\ell}|z_n|^2 dxdt + Cs^{d-3}\sum_{i=1}^{n} \iint\limits_{\Omega\times(0,T)} e^{-2s\alpha}\xi(t)^{d-3}|f_i - \tilde{f}_i|^2 dxdt \\ &\quad + Cs^{d-3}\sum_{i=1}^{n} \iint\limits_{\Omega\times(0,T)} e^{-2s\alpha}\xi(t)^{d-3}|g_i(Y) - g_i(\tilde{Y})|^2 dxdt, \end{aligned} \tag{5.6}$$

where C is a positive constant independent of F and $\tilde{F}$.

Now, from hypothesis H11 we deduce

$$s^{d-3}\sum_{i=1}^{n} \iint\limits_{\Omega\times(0,T)} e^{-2s\alpha}\xi(t)^{d-3}|g_i(Y) - g_i(\tilde{Y})|^2 dxdt \leq Cs^{d-3}\sum_{i=1}^{n} \iint\limits_{\Omega\times(0,T)} e^{-2s\alpha}\xi(t)^{d-3}|U|^2 dxdt.$$

Thus, for any $s > 1$, the last term in the right–hand side of (5.6) can be absorbed into the left–hand side, and therefore for s large enough, (5.6) can be written as follows:

$$\sum_{i=1}^{n} \mathcal{I}(3(n+1-i), z_i) \leq Cs^{\ell} \iint\limits_{\omega\times(0,T)} e^{-2s\alpha}\xi(t)^{\ell}|z_n|^2 dxdt + Cs^{d-3}\sum_{i=1}^{n} \iint\limits_{\Omega\times(0,T)} e^{-2s\alpha}\xi(t)^{d-3}|f_i - \tilde{f}_i|^2 dxdt. \tag{5.7}$$

On the other hand, noting that $e^{-2s\alpha(x,0)} = 0$ for $x \in \overline{\Omega}$, for $z_i \in H^1(0,T;L^2(\Omega))$, $i = 1,\dots,n$, we have

$$
\begin{aligned}
& s^{d-2} \int_\Omega \xi^{d-3}(T')|z_i(x,T')|^2 e^{-2s\alpha(x,T')}dx \\
&= s \int_0^{T'} \frac{\partial}{\partial t}\left(\int_\Omega (s\xi)^{d-3}|z_i|^2 e^{-2s\alpha} dx \right) dt \\
&= s^{d-2} \int_\Omega \int_0^{T'} \left((\partial_t \xi^{d-3})|z_i|^2 + 2\xi^{d-3} z_i \partial_t z_i - 2s\xi^{d-3}(\partial_t \alpha)|z_i|^2 \right) e^{-2s\alpha} dxdt \\
&\leq C \iint_{\Omega\times(0,T')} \left((s\xi)^{d-2}|z_i|^2 + (s\xi)^{d-4}|\partial_t z_i|^2 + s^d \xi^{d-2}|z_i|^2 + (s\xi)^{d-1}|z_i|^2 \right) e^{-2s\alpha} dxdt.
\end{aligned}
\tag{5.8}
$$

Here we used the fact that $|\partial_t \alpha(x,t)| \leq C\xi^2(t)$ and $|\partial_t \xi(t)| \leq C\xi^2(t)$ for $(x,t) \in \Omega \times (0,T)$, and Young's inequality (i.e., $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, $\frac{1}{p} + \frac{1}{q} = 1, a, b > 0$) with $a = s^{\frac{d}{2}} \xi^{\frac{d-2}{2}} |z_i|$, $b = (s\xi)^{\frac{d-4}{2}} |\partial_t z_i|$ and $p = q = 2$.

Since $Y(\cdot,T') = \tilde{Y}(\cdot,T')$ (see assumption H13) and $G \in W^{1,\infty}(\mathbb{R}^n)^n$ (see assumption H11), we have $Z(\cdot,T') = \partial_t \sigma(T')(F(\cdot) - \tilde{F}(\cdot))$. Thus, using H12 as well as the inequalities (5.7), (5.8) and the fact that $d \leq 3$, we obtain

$$
\begin{aligned}
& s^{d-2} \sum_{i=1}^n \int_\Omega |f_i(x) - \tilde{f}_i(x)|^2 e^{-2s\alpha(x,T')} dx \\
&\leq Cs^{d-2} \int_\Omega \xi^{d-3}(T')|\partial_t \sigma(T')(F(x) - \tilde{F}(x))|^2 e^{-2s\alpha(x,T')} dx \\
&\leq Cs^{d-2} \sum_{i=1}^n \int_\Omega \xi^{d-3}(T')|z_i(x,T')|^2 e^{-2s\alpha(x,T')} dx \\
&\leq Cs^\ell \iint_{\omega\times(0,T)} e^{-2s\alpha} \xi(t)^\ell |z_n|^2 dxdt + Cs^{d-3} \sum_{i=1}^n \iint_{\Omega\times(0,T)} e^{-2s\alpha} \xi(t)^{d-3} |f_i(x) - \tilde{f}_i(x)|^2 dxdt.
\end{aligned}
\tag{5.9}
$$

Due to that $\alpha(x,T') \leq \alpha(x,t)$ for $(x,t) \in \Omega \times (0,T)$ and $d \leq 3$, taking $s > 1$ we can absorb the second term on the right–hand side onto the left–hand side. Therefore the proof of Theorem 4 is complete. □

Remark 9. *Recall that in* [16] *the authors have used Carleman estimates with two parameters (s and λ) to prove an inverse coefficient problem for a 2×2–order nonlinear system as* (1.3) *by data of only one component. It might be useful to adapt the sketch of the proof of Theorem 4 in order to extend* [16, Theorem 1.3] *for the n–dimensional case and considering nonlinearity terms in several equations. Since our purpose is to study inverse problems related to sources instead of inverse coefficient problems, we omite these details and we only remark this alternative.*

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